

A VARIATIONAL APPROACH TO A CLASS OF RESONANCE PROBLEMS WITH APPLICATION TO A WAVE EQUATION PROBLEM

Inaugural-Dissertation

zur Erlangung
der philosophischen Doktorwürde

vorgelegt der
Philosophischen Fakultät II
der Universität Zürich

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HELMUT H. W. HOFER
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Begutachtet von Herrn Prof. Dr. P. Hess

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Helmut Hofer

I. Introduction

Let $(H, (\cdot, \cdot))$ be a real Hilbert space and $\phi \in C^1(H, \mathbb{R})$. We impose the following hypotheses on ϕ :

(ϕ) The function $\phi \in C^1(H, \mathbb{R})$ has a gradient ϕ' admitting the decomposition $\phi'(u) = Au + Bu$, where A is a bounded linear selfadjoint operator and B a continuous potential operator having bounded range. Let $b \in C^1(H, \mathbb{R})$ be the potential of B , normalized by $b(0) = 0$. Suppose H admits the orthogonal decomposition $H = X \oplus Y \oplus Z$ with X, Y, Z being A -invariant closed subspaces and $1 \leq \dim Y < +\infty$. Moreover suppose that the potential $a : H \rightarrow \mathbb{R}$ of $A : a(u) = \frac{1}{2} (Au, u)$ verifies (with some $\alpha > 0$)

$$a(x) \geq \alpha \|x\|^2 \quad \forall x \in X$$

$$a(y) = 0 \quad \forall y \in Y$$

$$a(z) \leq -\alpha \|z\|^2 \quad \forall z \in Z.$$

Let $\phi_h(u) = \phi(u) + (h, u)$ and assume that the following variant of a very weak Palais-Smale condition holds: $\phi_h(u_n) \rightarrow c \in \mathbb{R}$, $\phi'_h(u_n) \rightarrow 0$ for a bounded sequence $(u_n) \subset H$ implies that c is a critical level of ϕ_h , i.e. $\exists u \in H$ with $\phi'_h(u) = 0$ and $\phi_h(u) = c$.

Define the set $\Theta \subset H$ by

$$\Theta = \{h \in H \mid \liminf (b(y_n) + (h, y_n)) = +\infty \text{ for all sequences } (y_n) \subset Y, \|y_n\| \rightarrow +\infty\},$$

and the positively homogeneous functional $J_b : Y \rightarrow \mathbb{R}$ by

$$J_b(y) = \liminf_{t \rightarrow +\infty} b(ty)/t.$$

Then J_b is lipschitz-continuous with constant

$$\|B\| = \sup \{\|Bu\| \mid u \in H\} \quad (< +\infty). \quad \text{In fact,}$$

$b(ty)/t \leq b(ty')/t + \|B\| \|y - y'\|$, from which we infer

$J_b(y) \leq J_b(y') + \|B\| \|y - y'\|$ for all $y, y' \in Y$, which proves our

assertion. Now we characterize the set Θ using J_b .

Lemma 1. (i) Θ is convex,

(ii) $\text{int}(\Theta) = \{h \in H \mid -(h, y) < J_b(y) \text{ for all } y \in Y \setminus \{0\}\}$

(iii) $\text{cl}(\Theta) \subset \{h \in H \mid -(h, y) \leq J_b(y) \text{ for all } y \in Y\}$.

(iv) If for all sequences $(y_n) \subset Y$, $\|y_n\| \rightarrow +\infty$,

we have $\liminf (b(y_n) - J_b(y_n)) = +\infty$, then Θ is

closed and $\Theta = \{h \in H \mid -(h, y) \leq J_b(y) \text{ for all } y \in Y\}$.

Denote by $P_- : H \rightarrow X$, $P : H \rightarrow Y$, $P_+ : H \rightarrow Z$ the orthogonal projections.

Theorem 1. Assume ϕ verifies (ϕ) and at least one of the operators P_-B or P_+B is compact. Suppose that for all sequences $(u_n) \subset H$ with $((I-P)u_n)$ bounded and $\|Pu_n\| \rightarrow +\infty$ the following holds

$$\liminf (b(u_n) - J_b(Pu_n)) = +\infty$$

$$\liminf (b(u_n) - (Bu_n, u_n)) = +\infty$$

(Thus Θ is closed by Lemma 1(iv)). Then there exists an open set U containing Θ such that

- (i) For all $h \in U$ ϕ_h possesses at least one critical point.
- (ii) For all $h \in U \setminus \Theta$ ϕ_h possesses at least two critical points.

The next theorem studies the case in which Θ is empty.

Theorem 2. Assume ϕ verifies (ϕ) and P_+B is compact. Moreover suppose that for all sequences $(u_n) \subset H$ with $((I-P)u_n)$ bounded and $\|Pu_n\| \rightarrow +\infty$ we have

$$\lim Bu_n = 0,$$

$$\lim b(u_n) = \Delta, \text{ where } \Delta \in (0, +\infty) \text{ is independent of the chosen sequence,}$$

$$b(u) < \Delta \text{ for all } u \in H.$$

Then there exists an open neighbourhood U of $X \oplus Z$ such that

- (i) For all $h \in \text{cl}(U)$, ϕ_h possesses at least one critical point.
- (ii) For all $h \in U \setminus X \oplus Z$, ϕ_h possesses at least two critical points.

(iii) The set $\hat{U} = \{h \in H \mid \Phi_h \text{ possesses a critical point}\}$ is closed.

Remark. If $\Delta = +\infty$ the same result holds. Observe that in that case $\Theta = X \otimes Z$ because $J_b(y) = \lim_{t \rightarrow +\infty} b(ty)/t = \lim_{t \rightarrow +\infty} \int_0^1 (B(t\tau y), y) d\tau$
 $= \int_0^1 \lim_{t \rightarrow +\infty} (B(t\tau y), y) d\tau = 0$ for $y \in Y$ and the assertion

follows from Lemma 1(iv).

Now we apply the results to a wave equation problem. Consider

$$(1) \begin{cases} u_{tt} - u_{xx} = cu + g(u) + h & \text{in } (0, \pi) \times \mathbb{R} \\ u(0, t) = u(\pi, t) = 0 & \text{for all } t \in \mathbb{R} \\ u \text{ is } 2\pi\text{-periodic in } t. \end{cases}$$

Here $c > 0$ an eigenvalue of $\square = \partial_t^2 - \partial_x^2$. Let C_∞ denote the real vector space of arbitrarily often differentiable functions $(x, t) \in (0, \pi) \times \mathbb{R}$, which are 2π -periodic in $t \in \mathbb{R}$ and verify $u(0, t) = u(\pi, t) = 0$ for all $t \in \mathbb{R}$. By H we denote the completion of C_∞ with respect to the norm $\|u\| = (u, u)^{1/2}$, $(u, v) = \int_M uv$ $M = (0, \pi) \times (0, 2\pi)$. Clearly $H \cong L^2((0, \pi) \times (0, 2\pi)) = L^2(M)$. H possesses a Hilbert basis consisting of eigenfunctions of $\square$, $(\psi_{nk})_{(n,k) \in \mathbb{N} \times \mathbb{Z}}$ with corresponding eigenvalues $n^2 - k^2$. Hence $\square \psi_{nk} = (n^2 - k^2) \psi_{nk}$.

Definition 1. Given $h \in H$, a function $u \in H$ is said to be a weak solution of

$$\begin{aligned}
 (2) \quad & \square u = h \quad \text{in } (0, \pi) \times \mathbb{R} \\
 & u(0, t) = u(\pi, t) = 0 \quad \text{for all } t \in \mathbb{R} \\
 & u \text{ is } 2\pi\text{-periodic in } t,
 \end{aligned}$$

if $(u, \square \gamma) = (h, \gamma)$ for all $\gamma \in C_\infty$.

With this definition, $\square$ induces a selfadjoint linear operator in H , again denoted by $\square$, having spectrum $\sigma(\square) = \{n^2 - k^2 \mid n \in \mathbb{N}, k \in \mathbb{Z}\}$ (see [2], [5], or [6]). Let $L = \square - c \text{ Id}$, $c > 0$ an eigenvalue of $\square$.

We have

(L) $L : H \supset D(L) \rightarrow H$ is a selfadjoint linear operator with a nontrivial kernel $N(L)$, $\dim N(L) < +\infty$, such that the functions in $N(L)$ have the unique continuation property, i.e. if $u \in N(L)$ and $\text{meas}(\{x \in M \mid u(x) = 0\}) > 0$, then $u \equiv 0$. Moreover $\sigma(L) \setminus \{-c\}$ contains only in $\mathbb{R}$ isolated eigenvalues having finite multiplicity.

Assume

(F0) $g : \mathbb{R} \rightarrow \mathbb{R}$ is continuous and has bounded range.

Moreover $s \rightarrow g(s) + cs$ is nondecreasing on $\mathbb{R}$.

g induces a continuous potential operator $F : H \rightarrow H : (Fu)(x, t) = g(u(x, t))$ having bounded range and a normalized potential $f \in C^1(H, \mathbb{R})$ defined by

$$f(u) = \int_M G \circ u, \quad G(s) = \int_0^s g(\tau) \, d\tau.$$

Observe that the weak solutions of (1) are exactly the solutions of the operator equation $Lu = Fu + h$.

In order to apply Theorem 1 we impose the following hypotheses on g

$$(F1) \quad (i) \quad \gamma_{\pm\infty} = \lim_{s \rightarrow \pm\infty} \pm G(s)/s \quad \text{exist in } \mathbb{R}^+$$

$$(ii) \quad \liminf_{s \rightarrow \pm\infty} (G(s) - \gamma_{\pm\infty} |s|) = +\infty$$

$$(iii) \quad \liminf_{|s| \rightarrow +\infty} (G(s) - g(s)s) = +\infty.$$

Observe that (i) and (ii) imply that $G(s) \rightarrow +\infty$ for $|s| \rightarrow +\infty$.

Let

$$\Theta = \{h \in H \mid - (h, y) \leq \gamma_{+\infty} \int_M y_+ + \gamma_{-\infty} \int_M y_- \quad \forall y \in N(L)\}$$

where $y_+ = \max(0, y)$, $y_- = -\min(0, y)$. Note that Θ is closed and convex.

We have the following existence and multiplicity result

Theorem 3. Assume g verifies (F0) and (F1). Then there exists an open set $\mathcal{U}$ containing Θ such that

- (i) For all $h \in \mathcal{U}$ (1) possesses at least one weak solution.
- (ii) For all $h \in \mathcal{U} \setminus \Theta$ (1) possesses at least two weak solutions.

Observe that all results remain true if g is depending on x, t under obvious changes in (F0) and (F1).

Theorem 3 extends existence and multiplicity results given in [6] for the wave equation problem. There it was assumed in contrast to (F1) that

$g_{\pm\infty} = \lim_{s \rightarrow \pm\infty} g(s)$ exist in $\mathbb{R}$ and $g_{-\infty} \leq 0 \leq g_{+\infty}$,

$g(s) \geq g_{+\infty}$, $g(s) \leq g_{-\infty}$ for all $s \geq s_0$ and $s \leq -s_0$, respectively, for a suitable constant $s_0 > 0$, and

$$\liminf_{s \rightarrow \pm\infty} (g(s) - g_{\pm\infty})s > 0.$$

Observe that here it is not necessary that $g_{\pm\infty}$ exist, and moreover there are examples in which the second hypothesis is not verified but (F1) holds. Further the third hypothesis in [6] implies that $g(s)$ tends to $g_{\pm\infty}$ not more rapidly than $\frac{1}{|s|}$. For example $g(s) = 2s((\ln(1+s^2))(1+s^2))^{-1}$ verifies (F1) but not the hypothesis of [6] ($G(s) = \ln(\ln(1+s^2))$).

Note however that the abstract results proved in [6] apply also in the nonvariational case and are obtained by degree theory in connection with an approximation procedure, in contrast to the variational method used here. Next we study the case where G has bounded range. We impose the following hypotheses on g :

$$(F2) \quad (i) \quad \lim_{s \rightarrow \pm\infty} g(s) = 0.$$

$$(ii) \quad \lim_{s \rightarrow +\infty} G(s) = \lim_{s \rightarrow -\infty} G(s) = G_{\infty} \in \mathbb{R}^+.$$

$$(iii) \quad G_{\infty} < G(s) \text{ for all } s \in \mathbb{R}.$$

We have

Theorem 4. Assume g verifies (F0) and (F2). Then there exists

an open neighbourhood U of $N(L)^\perp$ such that

- (i) (1) possesses at least one weak solution for $h \in \text{cl}(U)$
- (ii) (1) possesses at least two weak solutions for
 $h \in U \setminus N(L)^\perp$
- (iii) $R(L-F)$ is closed.

Results similar to Theorem 3 and Theorem 4 hold for (variational) elliptic problems without monotonicity assumption on $g + c \text{Id}_{\mathbb{R}}$.

In that case we obtain strong extensions of results given in [4] and [7]. For example we can treat the problem

$-\Delta u = \lambda_n u + g(u) + h$ in Ω , $u = 0$ on $\partial\Omega$ where $g(s) = s(1+|s|^\sigma)^{-1}$, $\sigma > 1$, which could not be treated in [4] for $\sigma > 2$ and could be studied in [7] only for $\sigma \leq 3$ and $\lambda_n = \lambda_1$. Clearly Theorem 3 is also an extension of some results given in [6].

The results concerning the wave equation problem will be easy consequences of Theorem 1 and 2. Theorem 1 and 2 will be proved using the technical tool which will be developed in the following chapters II and III. An important role play a new deformation lemma and a perturbation theorem which says that for a small perturbation $\Psi \in C^1(H, \mathbb{R})$ of the functional $\Phi \in C^1(H, \mathbb{R})$, $\Phi + \Psi$ possesses critical points if Φ verifies a weak form of the Palais-Smale-condition and has a critical point given by a min-max characterisation of a special type, without knowing whether $\Psi + \Phi$ verifies the PS.-condition.

II. Some Theory of Flows, Intersections, and a Perturbation Theorem.

1. Deformations and Flows

We begin by proving some deformation lemmas. We show that the flows associated to a large class of variational problems in Hilbert spaces admit a special representation which is useful in order to study indefinite functionals. A first step in this direction was done by Benci and Rabinowitz, [1]. They constructed deformations having special representations by solving appropriate differential equations approximately by time-discretisation. Obviously this method has some disadvantages because one needs strong hypotheses in order to get the approximations uniformly on bounded sets. Here we obtain the flows and deformations under weaker hypotheses by exactly solving the appropriate differential equations.

Let us first introduce some notations.

For Ω , Y metric spaces, let

$$C(\Omega, Y) = \{f : \Omega \rightarrow Y \text{ continuous}\}$$

$$C_b(\Omega, Y) = \{f \in C(\Omega, Y) \mid f \text{ maps bounded sets into bounded sets}\}$$

$$K(\Omega, Y) = \{f \in C_b(\Omega, Y) \mid f \text{ maps bounded sets into relatively compact sets}\}.$$

Let $(H, (\cdot, \cdot))$ be a real Hilbert space and

$$L(H) = \{A : H \rightarrow H \mid A \text{ a bounded linear operator}\}$$

$$G(H) = \{A \in L(H) \mid A \text{ is bijective}\}.$$

For a given selfadjoint operator $A \in L(H)$ introduce

$[A]$ = smallest closed subalgebra of $L(H)$ containing
 A and Id , and

$$G_A(H) = G(H) \cap [A]$$

equipped with the induced metric from $L(H)$. Note that $[A]$ and $G_A(H)$ are commutative. Let $Q : H \rightarrow H$ be an orthogonal projection commuting with A and $\Omega \subset X \times H$ where X is a metric space. Denote by $\pi : \Omega \rightarrow H : (x, h) \rightarrow h$ the projection. We define a set $M(A, Q, \Omega)$ by

$$M(A, Q, \Omega) = \{ \eta : \Omega \rightarrow H \mid \eta \text{ admits the representation}$$

$$\eta(\omega) = \gamma(\omega)(\pi\omega) + K(\omega) \text{ with}$$

$$\gamma \in K(\cdot, G_A(H)), K \in C_b(\cdot, H)$$

$$QK \in K(\cdot, H) \}.$$

Let $\phi \in C^1(H, \mathbb{R})$ and $c, c' \in \mathbb{R}$. Denote by ϕ_c , $\phi^{c'}$, and $\phi_c^{c'}$ the sets $\phi^{-1}([c, +\infty))$, $\phi^{-1}((-\infty, c'])$ and $\phi_c \cap \phi^{c'}$, respectively.

Under suitable hypotheses on ϕ' we first prove the existence of flows of class $M(A, Q, \mathbb{R} \times H)$ having some useful properties.

Proposition 1. Assume $\phi \in C^1(H, \mathbb{R})$ and let the gradient ϕ' have the decomposition $\phi'(u) = Au + Bu$, where $A \in L(H)$ is a selfjoint linear operator and $B \in C(H, H)$. Further let $Q : H \rightarrow H$ be an orthogonal projection commuting with A such that $QB \in K(H, H)$. Then for arbitrarily given open sets V, W with $\text{cl}(V) \cap \text{cl}(W) = \emptyset$,

and $\|\phi'(u)\| \geq \varepsilon > 0$ for all $u \in V$ there exists a flow $\eta \in M(A, Q, \mathbb{R} \times H)$, $\eta(t, u) = \gamma(t, u)u + K(t, u)$ with $K(0, u) = 0$, $\gamma(0, u) = \text{Id}$ for all u , which is continuously differentiable with respect to $t \in \mathbb{R}$, such that

$$(i) \quad \frac{d}{dt} \Phi(\eta(t, u)) \geq \frac{1}{2} \beta(\eta(t, u)) \|\phi'(\eta(t, u))\| \text{ for all}$$

$(t, u) \in \mathbb{R} \times H$ where $r : H \rightarrow [0, 1]$ is a continuous function with $\beta|_W \equiv 0$ and $\beta|_V \equiv 1$.

$$(ii) \quad \eta(t, u) = u \quad \text{for all } (t, u) \in \mathbb{R} \times \text{cl}(W)$$

$$(iii) \quad \|u\| e^{-|t|} \leq \|\eta(t, u)\| \leq (\|u\| + 1) e^{|t|} \text{ for all } (t, u) \in \mathbb{R} \times H.$$

As a corollary we obtain a deformation lemma which states in addition to the usual properties that the deformation is in $M(A, Q, \mathbb{R} \times H)$ under the hypotheses of Proposition 1.

Corollary 1. Assume Φ and Q are as in Proposition 1. If there are numbers $d_1, d_2 \in \mathbb{R}$, $d_1 \leq d_2$ and $\varepsilon > 0$ such that $\|\phi'(u)\| \geq \varepsilon$ for all $u \in \Phi_{d_1}^{d_2}$ and if U is a given open set containing $\Phi_{d_1}^{d_2}$, then there exists a flow $\eta \in M(A, Q, \mathbb{R} \times H)$ with $\gamma(0, u) = \text{Id}$, $K(0, u) = 0$, which is continuously differentiable with respect to the first variable and satisfies

$$(i) \quad \frac{d}{dt} \Phi(\eta(t, u)) \geq 0 \quad \text{for all } (t, u) \in \mathbb{R} \times H$$

$$(ii) \quad \eta(1, \Phi_{d_1}^{d_2}) \subset \Phi_{d_2}^{d_2}$$

$$(iii) \quad \eta(t, \cdot)|_{H \setminus U} = \text{Id} \quad \text{for all } t \in \mathbb{R}.$$

Corollary 2. Under the hypotheses of Corollary 1, the level set $\Phi_{d_2}^{d_2}$ is a strong deformation retract of $\Phi_{d_1}^{d_2}$ with retraction map r in $M(A, Q, [0, 1] \times \Phi_{d_1}^{d_2})$, such that $r(\sigma, u) = \gamma(\sigma, u)u + K(\sigma, u)$ verifies $\gamma(0, u) = \text{Id}$, $K(0, u) = 0$ and for all $u \in \Phi_{d_1}^{d_2}$ the map $\sigma \rightarrow \Phi(r(\sigma, u))$ is non-decreasing. Moreover r is the restriction of map $\tilde{r} \in M(A, Q, [0, 1] \times H)$.

Corollary 1 and 2 together with the following Theorem 5 are a useful tool to prove the existence of critical points of $\Phi \in C^1(H, \mathbb{R})$. It should be remarked that they are in principle well-known except for the very useful statement that η and r can be chosen in $M(A, Q, \cdot)$.

2. Intersections

Let $M^*(A, Q) = \{(\eta, \tilde{\eta}) \mid \eta \in M(A, Q, [0, 1] \times H), \tilde{\eta} \in M(A, \text{Id}, [0, 1] \times H), \eta(t, u) = \gamma(t, u)u + K(t, u) \text{ with } \gamma(0, u) = \text{Id}, K(0, u) = 0 \text{ for all } u \in H \text{ and similarly for } \tilde{\eta}\}$. Let $\tilde{M} \subset M^*(A, Q)$ be a nonempty subset such that $(\eta_0, \tilde{\eta}_0) \in \tilde{M}$, where $\eta_0(t, u) = u$ for all $(t, u) \in [0, 1] \times H$.

Definition 2. Let $U_1 \subset U_2$ and $V_1 \subset V_2$ be subsets of H such that $V_1 \cap U_2 = V_2 \cap U_1 = \emptyset$. We say that (U_1, U_2) and (V_1, V_2) intersect nontrivially with respect to $\tilde{M}$, if for all $(\eta, \tilde{\eta}) \in \tilde{M}$ we have:

$$\begin{aligned} & \text{If } \eta(t, U_1) \cap \tilde{\eta}(t, V_2) = \emptyset \text{ and } \eta(t, U_2) \cap \tilde{\eta}(t, V_1) = \emptyset \\ & \text{for all } t \in [0, 1] \end{aligned}$$

$$\Rightarrow \eta(t, U_2) \cap \tilde{\eta}(t, V_2) \neq \emptyset \text{ for all } t \in [0, 1].$$

In that case we write $(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$.

Observe that in general $(U_1, U_2) \stackrel{\tilde{M}}{\sim} (V_1, V_2)$ and $(V_1, V_2) \stackrel{\tilde{M}}{\sim} (U_1, U_2)$ are different statements and there are examples in $H = \mathbb{R}^3$ where $(U_1, U_2) \stackrel{M}{\sim} (V_1, V_2)$ but $(V_1, V_2) \not\stackrel{M}{\sim} (U_1, U_2)$.

To motivate the definition let us give some examples in $H = \mathbb{R}^n$.

1. Let H_1 be a closed subspace of H and $H_2 = H_1^\perp$. Let $U \subset H_1$ and $V \subset H_2$ be relatively open bounded sets with $U \cap V = \{0\}$ ($\Rightarrow 0 \in U, 0 \in V$). Define $U_1 = \partial U$, $U_2 = \text{cl}(U)$, $V_1 = \partial V$, $V_2 = \text{cl}(V)$, where ∂ denotes the relative boundary and let $\eta, \tilde{\eta} : [0, 1] \times H \rightarrow H$ be continuous deformations of the identity map such that

$$\eta(t, U_1) \cap \tilde{\eta}(t, V_2) = \emptyset$$

$$\eta(t, U_2) \cap \tilde{\eta}(t, V_1) = \emptyset$$

for all $t \in [0, 1]$. This implies

$$\eta(t, u) - \tilde{\eta}(t, v) \neq 0 \quad \text{for all } (u, v) \in \partial(U \times V), \quad t \in [0, 1]$$

(∂ taken in $H_1 \times H_2$). Denote by $P : H \rightarrow H_1$ the orthogonal projection.

We infer

$$(P(\eta(t, u) - \tilde{\eta}(t, v)), (I - P)(\tilde{\eta}(t, v) - \eta(t, u))) \neq (0, 0)$$

$$\text{for all } (u, v) \in \partial(U \times V), \quad t \in [0, 1]$$

and deduce by the homotopy - invariance of the Brouwer - degree for the operator

$$T : [0, 1] \times H_1 \times H_2 \rightarrow H_1 \times H_2$$

$$T(t, u, v) = (P(\eta(t, u) - \tilde{\eta}(t, v)), (I - P)(\tilde{\eta}(t, v) - \eta(t, u)))$$

that $\deg(T(t, \cdot), U \times V, 0) = \deg(\text{Id}, U \times V, 0) = 1$, since $(0, 0) \in U \times V$.

This implies that $\eta(t, U) \cap \tilde{\eta}(t, V) \neq \emptyset$ or equivalently that

$(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$ where $\tilde{M}$ = Family of pairs consisting of all continuous perturbations of the identity.

2. Next a more difficult example in $H = \mathbb{R}^n$. Assume $H = H_1 \oplus H_2 \oplus H_3$ where the H_i are mutually orthogonal subspaces, and $\dim H_2 = 1$. Let $H_2 = \mathbb{R}\alpha$, $\|\alpha\| = 1$. Define $U_1 = \emptyset$, $U_2 = \{u \in H_1 \oplus H_2 \mid \|u\| = 1\}$ and $V_1 = H_3$, $V_2 = \mathbb{R}^+\alpha \oplus H_3$, and $\tilde{M} = \{(\eta, \eta_0) \mid \eta : [0, 1] \times H \rightarrow H \text{ continuous } \eta(0, u) = u \text{ for all } u \in H\}$. To show that

$(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$ is more difficult than the previous case

because one cannot formulate the problem directly as an operator equation of the form $T(t, u, v) = 0$ to which the Brouwer - degree is applicable. (Moreover observe that $(V_1, V_2) \overset{M}{\not\sim} (U_1, U_2)$).

But we can proceed as follows. Let $U = \{u \in H_1 \oplus H_2 \mid \|u\| < 2\}$

and for given $(\eta, \eta_0) \in \tilde{M}$ chose a constant $r > 0$ such that

$\eta([0, 1] \times \text{cl}(U)) \subset \{u \in H \mid \|u\| < r\} = \text{int}(B_r)$ and define

$\delta : [0, 1] \times (H_1 \oplus H_2) \times (H_2 \oplus H_3) \rightarrow H_2 : \delta(t, u, v) = (\|u\| - 1)\alpha$.

Clearly $\eta(t, U_2) \cap V_2 \neq \emptyset$ is equivalent to

$$\eta(t, u) - v = 0, \quad \delta(t, u, v) = 0,$$

for $t \in [0, 1]$ and $(u, v) \in \text{cl}(U) \times (B_r \cap (\mathbb{R}^+\alpha \oplus H_3))$.

Now this can be formulated as an operator equation and solved

by making use of the Brouwer - degree as in the previous case. However it is more difficult. Here the following result for the general case is useful.

Theorem 5. Let $Q_i : H \rightarrow H_i$, $i=1,2,3$, be orthogonal projections commuting with A such that H admits the orthogonal decomposition $H = \oplus_i H_i$ and suppose $\dim H_2 < +\infty$. Let $U \subset H_1 \oplus H_2$ and $V \subset H_2 \oplus H_3$ be relatively open sets such that $U \cap V \neq \emptyset$, and let $J = \{0,1\}$. Assume $(\eta, \tilde{\eta}) \in M^*(A, Q_1 + Q_2)$ and let $\delta : J \times (H_1 \oplus H_2) \times (H_2 \oplus H_3) \rightarrow H_2$ be a continuous bounded map and define $\delta^* : H_2 \rightarrow H_2$ by $\delta^*(u) = \delta(0, u, u)$. Suppose there exist constants $\rho_1 > 0$, $\rho_2 > 0$ such that $\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2} \neq \emptyset$ and that

$$(\eta(t, u) - \tilde{\eta}(t, v), \delta(t, u, v)) \neq (0, 0) \text{ for all } t \in J$$

and all $(u, v) \in \partial(\overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2})$,

where $\overset{\circ}{U}_{\rho_1} = U \cap \text{int}(B_{\rho_1})$, $\overset{\circ}{V}_{\rho_2} = V \cap \text{int}(B_{\rho_2})$ and

∂ is taken in $(H_1 \oplus H_2) \times (H_2 \oplus H_3)$. Further suppose that

$$\deg(\delta^*, \overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2}, 0) \neq 0$$

(with the convention $\deg(\delta^*, \{0\}, 0) = 1$ if $\dim H_2 = 0$,

observe that $\deg(\delta^*, \overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2}, 0)$ is well-defined since

$\delta^*(u) \neq 0$ on $\partial_{H_2}(\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2})$ if $\dim H_2 \geq 1$)

Then for all $t \in J$ there exists $(u, v) \in U \times V$ with

$$\delta(t, u, v) = 0 \quad \text{and} \quad \eta(t, u) = \tilde{\eta}(t, v).$$

Now we use Theorem 5 to give examples of intersections in a general Hilbert space H .

Example 1. Let $W \subset QH$, $\tilde{W} \subset (I-Q)H$ be relatively open bounded sets with $W \cap \tilde{W} = \{0\}$ and let $\tilde{M} = M^*(A, Q)$. Define $U_2 = \text{cl}(W)$, $U_1 = \partial W$, $V_2 = \text{cl}(\tilde{W})$ and $V_1 = \partial \tilde{W}$ (relative boundaries). Then $(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$. In fact, apply Theorem 5 with $Q_1 = Q$, $Q_2 = 0$, $Q_3 = (I-Q)$, $\delta \equiv 0$ and $U = W$, $V = \tilde{W}$.

Example 2. Let $W, \tilde{W}, U_1, U_2, V_1, V_2$ be as above and let

$$\tilde{M} = \{(\eta, \eta_0) \in M^*(A, (I-Q)) \mid \eta = \hat{\eta}|_{[0,1] \times H}, \hat{\eta} \in M(A, I-Q, \mathbb{R} \times H)\}$$

is a flow with $\hat{\gamma}(0, u) = \text{Id}$, $\hat{K}(0, u) = 0$. Then $(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$

Observe that $\eta(t, U_2) \cap V_2 \neq \emptyset$ is equivalent to $\hat{\eta}(-t, V_2) \cap U_2 \neq \emptyset$.

Therefore $(U_1, U_2) \overset{\tilde{M}}{\sim} (V_1, V_2)$ is equivalent to $(V_1, V_2) \overset{\tilde{M}}{\sim} (U_1, U_2)$

since $(\hat{\eta}(-(\cdot), \cdot)|_{[0,1] \times H}, \eta_0) \in \tilde{M}$. Now apply Theorem 5 with

$$Q_1 = (I-Q), \quad Q_2 = 0, \quad Q_3 = Q, \quad \delta \equiv 0 \quad \text{and} \quad U = \tilde{W}, \quad V = W.$$

Example 3. Let $U_1 = \emptyset$, $U_2 = \{u \in QH \mid \|u\| = 1\}$,

$V_2 = [0, 2] \alpha \oplus \{u \in (I-Q)H \mid \|u\| \leq 1\}$ for some $\alpha \in QH$, $\|\alpha\| = 1$,

and $V_1 = \partial V_2$. Define $\tilde{M} = \{(\eta, \eta_0) \in M^*(A, Q)\}$. Then $(U_1, U_2) \overset{\tilde{M}}{\sim} (V_1, V_2)$.

To prove this let $U = \{u \in QH \mid \|u\| < 2\}$ and $V = (0, 2)\alpha \oplus \{u \in (I-Q)H \mid \|u\| < 1\}$

Let $Q_1 : H \rightarrow QH \cap \{\alpha\}^\perp$, $Q_2 : H \rightarrow \mathbb{R}\alpha$, $Q_3 = I - Q$ be the orthogonal projections and define $\delta : J \times (H_1 \oplus H_2) \times (H_2 \oplus H_3) \rightarrow H_2$ by $\delta(t, u, v) = (\|u\| - 1)\alpha$. Then $\deg(\delta^*, U \cap V, 0) = \deg(\text{Id} - \alpha, (0, 2)\alpha, 0) = \deg(\text{Id}, (0, 2)\alpha, \alpha) = 1$ and an application of Theorem 5 yields the result.

Example 4. Let U_1 and V_1 be as above and define

$\tilde{M} = \{(\eta, \eta_0) \in M^*(A, I - Q) \mid \eta = \hat{\eta} \mid [0, 1] \times H, \hat{\eta} \in M(A, I - Q, \mathbb{R} \times H)\}$ is a flow with $\hat{\gamma}(0, u) = \text{Id}$, $\hat{K}(0, u) = 0$. Then $(U_1, U_2) \stackrel{\tilde{M}}{\sim} (V_1, V_2)$.

First observe that $\eta(t, U_1) \cap V_2 \neq \emptyset$ is equivalent to

$\hat{\eta}(-t, V_2) \cap U_2 \neq \emptyset$ and that $(\hat{\eta}(-(\cdot), \cdot) \mid [0, 1] \times H, \eta_0) \in \tilde{M}$.

Let $Q_1 = I - Q$, $Q_2 : H \rightarrow \mathbb{R}\alpha$, $Q_3 : H \rightarrow QH \cap \{\alpha\}^\perp$ and define

$\delta : J \times (H_1 \oplus H_2) \times (H_2 \oplus H_3) \rightarrow H_2$ by $\delta(t, u, v) = (\|u\| - 1)\alpha$. Let $U = (0, 2)\alpha \oplus \{u \in (I - Q)H \mid \|u\| < 1\}$ and $V = \{u \in QH \mid \|u\| < 2\}$.

As above the result follows from Theorem 5.

Remark. Similar statements hold if we replace (U_1, U_2) and

(V_1, V_2) by (U_1^*, U_2^*) and (V_1^*, V_2^*) where $U_1^* = \eta(1, U_1)$, $V_1^* = \eta(1, V_1)$

where $\eta \in M(A, \text{Id}, [0, 1] \times H)$ with $\eta(t, u) = \gamma(t, u)u + K(t, u)$,

$\gamma(0, u) = \text{Id}$, $K(0, u) = 0$ for all $u \in H$, such that η is the

restriction of a flow $\hat{\eta} \in M(A, \text{Id}, \mathbb{R} \times H)$ i.e. $\eta = \hat{\eta} \mid [0, 1] \times H$.

For example the translation flow $\hat{\eta}(t, u) = u + t u_0$ for a fixed

$u_0 \in H$.

3. A Perturbation Result

Define $E = C_b^1(H, \mathbb{R})$ to be the vector space consisting of all $\Psi \in C^1(H, \mathbb{R})$ such that the map $u \rightarrow \Psi'(u)$ is bounded. Let τ_E be the topology of uniform convergence of Ψ and Ψ' on bounded sets. τ_E is metrizable. We say that a functional $\phi \in C^1(H, \mathbb{R})$ verifies the generalized Palais - Smale - Condition on the interval $[d_1, d_2]$ if

(GPS) $_{[d_1, d_2]}$ For all sequences $(u_n) \subset \phi_{d_1}^{d_2}$ with $\phi'(u_n) \rightarrow 0$ it follows that (u_n) is bounded.

We say that it verifies the weak Palais - Smale - Condition on $[d_1, d_2]$ if

(WPS) $_{[d_1, d_2]}$ For all sequences $(u_n) \subset \phi_{d_1}^{d_2}$ with $\phi'(u_n) \rightarrow 0$ and $\phi(u_n) \rightarrow d \in [d_1, d_2]$ it follows that d is a critical value of ϕ .

We have

Theorem 6. Let $\phi \in C^1(H, \mathbb{R})$ and assume ϕ' has the decomposition $\phi'(u) = Au + Bu$ with $A \in L(H)$ being a selfadjoint linear operator and $B \in C(H, H)$ such that $QB \in K(H, H)$ for some orthogonal projection Q commuting with A . Let $\tilde{M} = \{(\eta, \eta_0) \in M^*(A, Q) \mid \eta$ is the restriction of a flow $\hat{\eta} \in M(A, Q, \mathbb{R} \times H)\}$. Assume $(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$ (or $(V_1, V_2) \overset{M}{\sim} (U_1, U_2)$) where the sets U_i, V_i are bounded and

$$\inf \phi(U_1) > \sup \phi(V_2)$$

$$\inf \phi(U_2) > \sup \phi(V_1)$$

Then

(i) For given $\tilde{\epsilon} > 0$ there exists $W \in \mathcal{T}_E$, $O \in W$, such that for all $\Psi \in W$, $\Psi' = \bar{A} + \bar{B}$ with $\bar{A} \in [A]$ and $\bar{B} \in C_b(H, H)$, $Q\bar{B} \in K(H, H)$, there is a sequence $(u_n) \subset H$ with $(\phi + \Psi)(u_n) \rightarrow d$, $(\phi + \Psi)'(u_n) \rightarrow 0$, where $d \in [d_1, d_2]$, $d_1 = \inf \phi(U_2) - \tilde{\epsilon}$, $d_2 = \sup \phi(V_2) + \tilde{\epsilon}$.

(ii) If in addition ϕ verifies the (GPS)-condition on $[d_1, d_2]$ there exist a constant $r^* > 0$ and $W \in \mathcal{T}_E$, $O \in W$, such that for all $\Psi \in W$ with Ψ' having a representation as in (i) we have

$$\inf \{ \|(\phi + \Psi)'(u)\| \mid u \in (\phi + \Psi)_{d_1}^{d_2} \cap B_{r^*} \} = 0.$$

Observe that we only need the (GPS)-condition for ϕ but not for the perturbed problems. Note that all linear functionals $\psi_h : \psi_h(u) = (h, u)$ are in W for small $\|h\|$ and have an admissible representation.

III. Proof of the Abstract Results

Proof of Proposition 1. Since $QB : H \rightarrow QH$ is compact we find a locally lipschitz continuous compact mapping $B_1 : H \rightarrow QH$ such that

$$\|B_1 u - QBu\| \leq \epsilon/16 \quad \text{for all } u \in H,$$

and similarly a locally lipschitz continuous mapping

$B_2 : H \rightarrow (I-Q)H$ with

$$\|B_2 u - (I-Q)Bu\| \leq \epsilon/16 \quad \text{for all } u \in H.$$

Then we have for $\tilde{B} = B_1 + B_2$

$$\|\tilde{B}u - Bu\| \leq \epsilon/8 \quad \text{for all } u \in H,$$

and $Q\tilde{B}$ is compact. Let $C(u) = Au + \tilde{B}u$. For $u \in \text{cl}(V)$ we have

$\|C(u)\| \geq \frac{7}{8} \epsilon$. We find an open neighbourhood U of $\text{cl}(V)$ such

that $\text{cl}(U) \cap \text{cl}(W) = \emptyset$ and $\|C(u)\| \geq \frac{6}{8} \epsilon$ for all $u \in U$. There

exists a locally lipschitz continuous function $\beta : H \rightarrow [0,1]$

such that $\beta|_{\text{cl}(V)} \equiv 1$ and $\beta|_{H \setminus U} \equiv 0$. Define a locally

lipschitz continuous operator $G : H \rightarrow H$ by

$$(3) \quad Gu \equiv \begin{cases} \beta(u) \|Au + \tilde{B}u\|^{-1} (Au + \tilde{B}u) & \text{if } u \in U \\ 0 & \text{if } u \in H \setminus U. \end{cases}$$

Clearly this is well-defined since $\|Au + \tilde{B}u\| = \|C(u)\| \geq \frac{1}{2} \epsilon$

for all $u \in U$. We infer for $u \in U$ (since $\|\phi'(u)\| \geq \|C(u)\| - \epsilon/8$

$\geq 5\epsilon/8$ for $u \in U$)

$$\begin{aligned} (\phi'(u), Gu) &= \beta(u) \|C(u)\|^{-1} (\|\phi'(u)\|^2 - (\phi'(u), Bu - \tilde{B}u)) \\ &= \beta(u) (\|\phi'(u)\| + \epsilon/8)^{-1} (\|\phi'(u)\|^2 - \epsilon \|\phi'(u)\|/8) \\ &= \beta(u) (\|\phi'(u)\| + (1+\frac{1}{5}))^{-1} (\|\phi'(u)\|^2 (1 - \frac{1}{5})) \\ &= \beta(u) \frac{5}{6} \|\phi'(u)\|^{-1} \frac{4}{5} \|\phi'(u)\|^2 \\ &\geq \frac{1}{2} \beta(u) \|\phi'(u)\|. \end{aligned}$$

Clearly it remains true if $u \in H \setminus U$ because in that case $Gu = 0$ and $\beta(u) = 0$. Therefore

$$(4) \quad (\Phi'(u), Gu) \geq \frac{1}{2} \beta(u) \|\Phi'(u)\| \text{ for all } u \in H.$$

We solve the ordinary differential equation

$$(5) \quad \dot{\eta}_u = G\eta_u, \quad \eta_u(0) = u \text{ for } u \in H.$$

Since G is locally lipschitz continuous and $\|Gu\| \leq 1$ for all $u \in H$ we have by the basic existence theorem global existence on $\mathbb{R}$ and obtain a flow $\eta : \mathbb{R} \times H \rightarrow H$ by $\eta(t, u) \equiv \eta_u(t)$.

Moreover we obtain from (5) the estimate

$$\|u\| e^{-|t|} \leq \|\eta(t, u)\| \leq (\|u\| + 1) e^{|t|} \text{ for all } (t, u) \in \mathbb{R} \times H.$$

Hence we have established (iii). (i) follows from

$$\frac{d}{dt} \Phi(\eta(t, u)) \geq \frac{1}{2} \beta(\eta(t, u)) \|\Phi'(\eta(t, u))\|$$

and (ii) since $\dot{\eta}(t, u) = 0$, $\eta(0, u) = u$ for $u \in W$. It remains to show that $\eta \in M(A, Q, \mathbb{R} \times H)$. Define $\alpha \in C_b(\mathbb{R} \times H, \mathbb{R}^+)$ and

$$\alpha_A \in K(\mathbb{R}^2 \times H, G_A(H)) \text{ by}$$

$$\alpha(t, u) = \begin{cases} \|\Phi(\eta(t, u))\|^{-1} & \text{if } \eta(t, u) \in U \\ 0 & \text{if } \eta(t, u) \in H \setminus U \end{cases}$$

and

$$\alpha_A(t, s, u) = \exp \left(\left(\int_0^{t-s} \alpha(\tau + s, u) d\tau \right) A \right).$$

The compactness of α_A follows immediately from the continuity

of $\exp : [A] \rightarrow G_A(H)$ and the compactness of the map

$$\mathbb{R}^2 \times H \rightarrow \mathbb{R}A \subset [A] : (t, s, u) \rightarrow \left(\int_0^{t-s} \alpha(\tau, s, u) d\tau \right) (A),$$

since the map is of class C_b and the range is a subset of a onedimensional Banach space.

Let $B^* : \mathbb{R} \times H \rightarrow H$ be defined by $B^*(s, u) = \tilde{B}(\eta(s, u))$. Clearly B^* is continuous and $QB^* \in K(\mathbb{R} \times H, H)$ because $QB \in K(H, H)$ and $\eta \in C_b(\mathbb{R} \times H, H)$. Define $\chi : \mathbb{R} \times H \rightarrow H$ by

$$\begin{aligned} \chi(t, u) &= \alpha_A(t, 0, u)u + \int_0^t (\alpha_A(t, s, u) (\alpha(s, u) B^*(s, u))) ds \\ &= \gamma(t, u)u + K(t, u). \end{aligned}$$

Clearly $\gamma \in K(\mathbb{R} \times H, G_A(H))$ and QK is compact since Q and $\alpha_A(t, s, u)$ commute for all (t, s, u) and the map

$\mathbb{R}^2 \times H \rightarrow H : (t, s, u) \rightarrow \alpha_A(t, s, u) (\alpha(s, u) QB^*(s, u))$ is compact.

Hence $\chi \in M(A, Q, \mathbb{R} \times H)$. Moreover $\chi(\cdot, u)$ is continuously differentiable for fixed $u \in H$ and one easily verifies that it solves (5). By uniqueness $\chi = \eta$.

□

Proof of Corollary 1. There exists an open neighbourhood V

of $\phi_{d_1}^{d_2}$ such that $cl(V) \subset U$ and $\|\phi'(u)\| \geq \varepsilon/2$ for all $u \in V$.

Let $W = H \setminus U$. By Proposition 1 there exists a flow

$\in M(A, Q, \mathbb{R} \times H)$ with

$$\frac{d}{dt} \phi(\eta(t, u)) \geq 0 \quad \text{for all } (t, u) \in \mathbb{R} \times H,$$

$$\frac{d}{dt} \phi(\tilde{\eta}(t, u)) \geq \frac{\varepsilon}{2} \quad \text{for all } (t, u) \in \mathbb{R} \times H \text{ such that}$$

$$\tilde{\eta}(t, u) \in \phi_{d_1}^{d_2}$$

$$\tilde{\eta}(t, u) = u \quad \text{for all } (t, u) \in \mathbb{R} \times (H \setminus U) .$$

Hence $\tilde{\eta}(t, \cdot)|_{H \setminus U} = \text{Id}$. If $u \in \Phi_{d_1}^{d_2}$ we infer for $t \in \mathbb{R}^+$ and

$$\omega(t) = \{s \in \mathbb{R}^+ \mid \tilde{\eta}([0, s], u) \subset \Phi_{d_1}^{d_2}, s \leq t\} :$$

$$(6) \quad \Phi(\tilde{\eta}(t, u)) \geq d_1 + \frac{\varepsilon}{2} \text{ meas } (\omega(t))$$

Let $t_\infty = \frac{2}{\varepsilon} (d_2 - d_1)$. If $\text{meas}(\omega(t_\infty)) < t_\infty$ we infer from the

definition of $\omega(t)$ since $u \in \Phi_{d_1}^{d_2}$, using the monotonicity

of $t \rightarrow \Phi(\tilde{\eta}(t, u))$

$$\Phi(\tilde{\eta}(t_\infty, u)) > d_2$$

and if $\text{meas}(\omega(t_\infty)) = t_\infty$ by (6)

$$\Phi(\tilde{\eta}(t_\infty, u)) \geq d_2 .$$

We define a flow $\eta \in M(A, Q, \mathbb{R} \times H)$ by $\eta(t, u) = \tilde{\eta}(tt_\infty, u)$. Clearly η has the desired properties.

□

Proof of Corollary 2. By Proposition 1 there exists a flow

$\tilde{\eta} \in M(A, Q, \mathbb{R} \times H)$ such that $\frac{d}{dt} \Phi(\tilde{\eta}(t, u)) \geq \frac{\varepsilon}{2}$ for all

$(t, u) \in \mathbb{R} \times H$ with $\tilde{\eta}(t, u) \in \Phi_{d_1}^{d_2}$. Define for $u \in \Phi_{d_1}^{d_2}$ a

number $t(u)$ by $t(u) = \inf\{t \in \mathbb{R}^+ \mid \tilde{\eta}(t, u) \notin \Phi_{d_1}^{d_2}\}$. Clearly

$t(u) \leq \frac{2}{\varepsilon} (d_2 - d_1)$. Observe that for $u \in \Phi_{d_1}^{d_2}$ $t(u)$ is the unique

number with $\Phi(\tilde{\eta}(t(u), u)) = d_2$. From the uniqueness of $t(u)$

we obtain by a standard compactness argument the continuity of t .

Since $\Phi_{d_1}^{d_2}$ is a closed subset of H there exists a continuous extension $\bar{t} : H \rightarrow [0, \frac{2}{\epsilon} (d_2 - d_1)]$ of t . Define $\tilde{r}$ by

$$\tilde{r} : [0, 1] \times H \rightarrow H : \tilde{r}(\sigma, u) = \tilde{\eta}(\sigma \bar{t}(u), u).$$

Clearly $\tilde{r} \in M(A, Q, [0, 1] \times H)$ since $\tilde{\eta} \in M(A, Q, \mathbb{R} \times H)$ and $\bar{t} \in C_b(H, \mathbb{R})$. By construction we have $\tilde{r}(1, \Phi_{d_1}^{d_2}) \subset \Phi_{d_2}^{d_2}$. Since $\tilde{\eta}(t, u) = \tilde{\gamma}(t, u)u + \tilde{K}(t, u)$ with $\tilde{\gamma}(0, u) = \text{Id}$ and $\tilde{K}(0, u) = 0$ and the mapping $s \rightarrow \Phi(\tilde{\eta}(s, u))$ is nondecreasing for fixed $u \in H$ we find that $r = \tilde{r}|_{[0, 1] \times \Phi_{d_1}^{d_2}}$ has the desired properties.

□

Proof of Theorem 5. Clearly $\eta(t, u) - \tilde{\eta}(t, v) = 0$ and $\delta(t, u, v) = 0$ for some $(t, u, v) \in \Gamma \equiv J \times \text{cl}(\tilde{U}_{\rho_1}^0) \times \text{cl}(\tilde{V}_{\rho_2}^0)$ is equivalent to

$$Q_1(\eta(t, u) - \tilde{\eta}(t, v)) = 0$$

$$Q_3(\eta(t, v) - \eta(t, u)) = 0$$

$$Q_2(\eta(t, v) - \eta(t, u)) + \eta(t, u, v) = 0$$

$$Q_2(\eta(t, u) - \eta(t, v)) + \eta(t, u, v) = 0$$

for some $(t, u, v) \in \Gamma$. Let $\eta(t, u) = \gamma(t, u)u + K(t, u)$ and $\tilde{\eta}(t, v) = \tilde{\gamma}(t, v)v + \tilde{K}(t, v)$ and define $\bar{\gamma}$ and $\bar{\tilde{\gamma}} \in K(\mathbb{R} \times H, G_A(H))$ by $\bar{\gamma}(t, u) = (\gamma(t, u))^{-1}$ and $\bar{\tilde{\gamma}}(t, v) = (\tilde{\gamma}(t, v))^{-1}$. Hence the system is equivalent to

$$0 = u + \tilde{\gamma}(t, u) (Q_1 + Q_2) (K(t, u) - \gamma(t, v) Q_2 v - K(t, v) + \delta(t, u, v))$$

$$\equiv u + K^*(t, u, v)$$

$$0 = v + \tilde{\gamma}(t, v) (Q_2 + Q_3) (\tilde{K}(t, v) - \gamma(t, u) Q_2 u - K(t, u) + \delta(t, u, v))$$

$$\equiv v + K^{**}(t, u, v).$$

Note that $(Q_2 + Q_3)K$ is not compact and therefore K^{**} is not compact. $K^* : \Gamma \rightarrow H$ is compact and Γ is bounded. Hence $K^*\Gamma$ is relatively compact. Therefore there exists a compact subset $C' \subset \text{cl}(\tilde{U}_{\rho_1}^0)$ such that $u \in C'$ for all $u \in \text{cl}(\tilde{U}_{\rho_1}^0)$ which are solutions of $u + K^*(t, u, v) = 0$. Let $C = (J \times C') \cup (\{0\} \times \text{cl}(\tilde{U}_{\rho_1}^0)) \subset \mathbb{R} \times (H_1 \oplus H_2)$. Observe that $(Q_2 + Q_3)K|_C$ is compact because $J \times C'$ is compact and $K|_{\{0\} \times \text{cl}(\tilde{U}_{\rho_1}^0)} \equiv 0$. By a well-known extension - theorem there exists a compact extension $K' : J \times \text{cl}(\tilde{U}_{\rho_1}^0) \rightarrow H$ of the compact mapping $(Q_2 + Q_3)K|_C$. Hence the operator $K'' : \Gamma \rightarrow H : K''(t, u, v) = \tilde{\gamma}(t, v) (Q_2 + Q_3) (\tilde{K}(t, v) - \gamma(t, u) Q_2 u - K'(t, u) + \delta(t, u, v))$ is compact. We define the operator $T : \Gamma \rightarrow (H_1 \oplus H_2) \times (H_2 \oplus H_3)$ by

$$T(t, u, v) = (u + K^*(t, u, v), v + K''(t, u, v)).$$

Obviously the solutions $(t, u, v) \in \Gamma$ of $T(t, u, v) = 0$ are exactly the solutions of $\eta(t, u) - \tilde{\eta}(t, v) = 0$, $\delta(t, u, v) = 0$ in Γ .

Moreover $T(0, u, v) = (u - Q_2 v + \delta(0, u, v), v - Q_2 u + \delta(0, u, v))$, and by our hypotheses we have $T(t, u, v) \neq 0$ for all $t \in J$ and all $(u, v) \in \partial(\tilde{U}_{\rho_1}^0 \times \tilde{U}_{\rho_2}^0)$. By the homotopy - invariance - property of the Leray - Schauder - Degree we deduce

$$\alpha(t) = \deg(T(t, \cdot), \overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2}, 0) = \deg(T(0, \cdot), \overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2}, 0) = \alpha(0).$$

Observe that we have $\alpha(0) = 1$ if $Q_2 = 0$. So let us assume for the following $Q_2 \neq 0$. Define a homotopy $T^1 : J \times (H_1 \oplus H_2) \times$

$$(H_2 \times H_3) \rightarrow (H_1 \times H_2) \times (H_2 \times H_3) \text{ by}$$

$$T^1(t, u, v) = (Q_1 u + t Q_2(u-v) + \delta(0, t Q_1 u + Q_2 u, tv + (1-t) Q_2 u), v - Q_2 u \\ + t \delta(0, t Q_1 u + Q_2 u, tv + (1-t) Q_2 u)).$$

Clearly $T^1(1, u, v) = T(0, u, v)$ for $(u, v) \in \text{cl}(\overset{\circ}{U}_{\rho_1}) \times \text{cl}(\overset{\circ}{V}_{\rho_2})$ and

$T^1(0, u, v) = (Q_1 u + \delta^*(Q_2 u), v - Q_2 u)$. Assume $T^1(t, u, v) = 0$ for some $t \in J$ and $(u, v) \in \partial(\overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2})$. We deduce $Q_1 u = Q_3 v = 0$,

so that $u = Q_2 u$ and $v = Q_2 v$ and therefore we find

$$t(u-v) + \delta(0, u, tv + (1-t)u) = 0$$

$$v-u + t \delta(0, u, tv + (1-t)u) = 0.$$

This implies $(t^2 + 1)(u-v) = 0$, hence $u = v$ and therefore

$$0 = \delta(0, u, u) = \delta^*(u). \text{ Now } (u, v) = (u, u) \in (\partial(\overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2})) \cap (H_2 \times H_2).$$

$$\text{This yields } u \in ((\overset{\circ}{U}_{\rho_1}) \cap \text{cl}(\overset{\circ}{V}_{\rho_2})) \cup (\text{cl}(\overset{\circ}{U}_{\rho_1}) \cap (\overset{\circ}{V}_{\rho_2})) = \partial(\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2}),$$

where the last boundary is taken in H_2 . But therefore $\delta^*(u) = 0$ for some $u \in \partial(\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2})$ which was excluded by our hypotheses.

We infer by the homotopy - invariance - property of the degree that

$$\alpha(t) = \alpha(0) = \deg(T^1(0, \cdot), \overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2}, 0).$$

Define a set $\tilde{\Gamma} = (Q_1 \overset{\circ}{U}_{\rho_1} \oplus (\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2})) \times ((\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2}) \oplus Q_3 \overset{\circ}{V}_{\rho_2})$
 $\subset (H_1 \oplus H_2) \times (H_2 \oplus H_3)$. If $T^1(0, u, v) = 0$ for $(u, v) \in \text{cl}(\tilde{\Gamma})$

we find $(u, v) \in (\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2}) \times (\overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2})$. The same conclusion holds by the preceding discussion if we assume $T^1(0, u, v) = 0$ for $(u, v) \in \text{cl}(\overset{\circ}{U}_{\rho_1}) \times \text{cl}(\overset{\circ}{V}_{\rho_2})$. Making use of the excision

property of the degree we arrive at

$$\begin{aligned}\alpha(t) &= \deg(T^1(0, \cdot), \overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2}, 0) = \deg(T^1(0, \cdot), \tilde{\Gamma} \cap (\overset{\circ}{U}_{\rho_1} \times \overset{\circ}{V}_{\rho_2}), 0) \\ &= \deg(T^1(0, \cdot), \tilde{\Gamma}, 0).\end{aligned}$$

Hence

$$\begin{aligned}\alpha(t) &= \deg(T^1(0, \cdot), \tilde{\Gamma}, 0) \\ &= \deg(\text{Id}, Q_1 \overset{\circ}{U}_{\rho_1} \times Q_3 \overset{\circ}{V}_{\rho_2}, 0) \deg(T^2, \overset{\circ}{D} \times \overset{\circ}{D}, 0) \\ &= \deg(T^2, \overset{\circ}{D} \times \overset{\circ}{D}, 0) \quad ((0, 0) \in Q_1 \overset{\circ}{U}_{\rho_1} \times Q_3 \overset{\circ}{V}_{\rho_2})\end{aligned}$$

where $\overset{\circ}{D} = \overset{\circ}{U}_{\rho_1} \cap \overset{\circ}{V}_{\rho_2}$ and $T^2 : H_2 \times H_2 \rightarrow H_2 \times H_2$ is defined by

$$T^2(u, v) = (\delta^*(u), v - u),$$

Let $\rho > \max(\rho_1, \rho_2)$ and $\hat{C} = \{u \in H_2 \mid \|u\| < \rho\}$. By the excision - property we find

$$\alpha(t) = \deg(T^2, \overset{\circ}{D} \times \hat{C}, 0),$$

and define a homotopy $T^3 : [0, 1] \times \text{cl}(\overset{\circ}{D}) \times \text{cl}(\hat{C}) \rightarrow H_2 \times H_2$ by

$$T^3(t, u, v) = (\delta^*(u), v - tu).$$

We have $T^3(t, u, v) \neq 0$ for all $t \in [0, 1]$ and $(u, v) \in \partial(\mathring{D} \times \hat{C})$ since $\text{cl}(\mathring{D}) \subset \hat{C}$. Moreover $T^3(1, \cdot) = T^2$ and $T^3(0, \cdot) = \delta^* \times \text{Id}$. Hence

$$\begin{aligned} \alpha(t) &= \deg(T^3(0, \cdot), \mathring{D} \times \hat{C}, 0) \\ &= \deg(\delta^*, \mathring{D}, 0) \quad \deg(\text{Id}, \hat{C}, 0) \\ &= \deg(\delta^*, \mathring{D}, 0) \\ &\neq 0. \end{aligned}$$

This proves the Theorem. □

Proof of Theorem 6 (ii). We leave it to the reader to show that $(U_1, U_2) \tilde{M} (V_1, V_2)$ and $(V_1, V_2) \tilde{M} (U_1, U_2)$ are equivalent. By the $(\text{GPS})_{[d_1, d_2]}$ - condition we find $\varepsilon > 0$ and $r > 0$ such that $u \in \phi_{d_1}^{d_2} \setminus \mathring{B}_r$ implies $\|\phi'(u)\| \geq \varepsilon$. By increasing r if necessary, we may assume $U_2 \subset \mathring{B}_r$. Choose a number $r^* \in \mathbb{R}$, such that

$$\ln(r^*/(r+1)) \geq \frac{4}{\varepsilon} (d_2 - d_1).$$

Observe that $d_2 - d_1 \geq 2\tilde{\varepsilon}$ since $U_2 \cap V_2 \neq \emptyset$. Because U_i and V_i are bounded sets we find $W \in \mathcal{T}_E$, $0 \in W$ with

$$(7) \quad \sup \{ \|\Psi'(u)\| \mid u \in B_{r^*} \} \leq \frac{\varepsilon}{2}$$

$$(8) \quad (\phi + \Psi)_{d_1 + \frac{1}{4}\tilde{\varepsilon}}^{d_2 - \frac{1}{4}\tilde{\varepsilon}} \cap B_{r^*} \subset \phi_{d_1}^{d_2} \cap B_{r^*}$$

$$(9) \quad d_1 + \frac{3}{4}\tilde{\varepsilon} \leq \inf(\phi + \Psi)(U_2), \quad \sup(\phi + \Psi)(V_2) \leq d_2 - \frac{3}{4}\tilde{\varepsilon}$$

$$(10) \quad \inf(\phi+\psi)(U_2) > \sup(\phi+\psi)(V_1) \\ \inf(\phi+\psi)(U_1) > \sup(\phi+\psi)(V_2)$$

for all $\psi \in W$. Now let $\psi \in W$ have the additional properties from our hypotheses and consider $\rho = \phi + \psi$. Clearly ρ' admits the decomposition $\rho' = (A + \bar{A}) + (B + \bar{B})$ with $A + \bar{A} \in [A]$, $B + \bar{B} \in C(H, H)$, $Q(B + \bar{B}) \in K(H, H)$. Assume

$$\alpha = \inf\{\|\rho'(u)\| \mid u \in B_{r*} \cap \rho_{d_1}^{d_2}\} > 0$$

By Proposition 1 we find a flow $\chi \in M(A, Q, \mathbb{R} \times H)$ such that

$$(11) \quad \frac{d}{dt} \rho(\chi(t, u)) \geq 0 \quad \text{for all } (t, u) \in \mathbb{R} \times H \\ \frac{d}{dt} \rho(\chi(t, u)) \geq \frac{1}{2} \|\rho'(\chi(t, u))\| \quad \text{for all } (t, u) \in \mathbb{R} \times H \\ \text{such that } \chi(t, u) \in B_{r*} \cap \rho_{d_1}^{d_2}, \\ \chi(t, u) = u \quad \text{for all } (t, u) \in \mathbb{R} \times (H \setminus \mathring{B}_{r*+1}), \\ \|\chi(t, u)\| \leq (\|u\| + 1) e^{|t|} \quad \text{for all } (t, u) \in \mathbb{R} \times H.$$

Define

$$C(\psi) = \liminf_{t \rightarrow +\infty} \rho(\chi(t, U_2)).$$

Observe that we have

$$\chi(t, U_2) \cap V_1 = \emptyset, \quad \chi(t, U_1) \cap V_2 = \emptyset \quad \text{for all } t \in \mathbb{R}^+$$

by (10) and (11). Now let $t_0 \geq 0$. Then the mapping

$$\eta : [0, 1] \times H \rightarrow H : (\sigma, u) \mapsto \chi(\sigma t_0, u) \text{ is such that } (\eta, \eta_0) \in \dot{M}.$$

Therefore

$$\chi(t_0, U_2) \cap V_2 \neq \emptyset \quad \text{for all } t_0 \geq 0$$

by our hypotheses. Hence

$$(12) \quad C(\Psi) \leq \sup \rho(V_2) \leq d_2 - \frac{3}{4} \tilde{\varepsilon}, \text{ and} \\ C(\Psi) \geq \inf \rho(U_2) \geq d_1 + \frac{3}{4} \tilde{\varepsilon}.$$

Let $t_\infty = \frac{2}{\alpha} (d_2 - d_1)$. Fix an arbitrarily chosen $u \in U_2$ and

assume first that we have for all $t \in [0, t_\infty]$ that

$$\chi(t, u) \in \rho_{d_1}^{d_2} \cap B_{r^*}. \quad \text{Then}$$

$$\begin{aligned} \rho(\chi(t_\infty, u)) &\geq d_1 + \frac{3}{4} \tilde{\varepsilon} + \int_0^{t_\infty} \frac{d}{dt} \rho(\chi(t, u)) dt \\ &= d_1 + \frac{3}{4} \tilde{\varepsilon} + \frac{1}{2} \frac{2}{1} (d_2 - d_1) \\ &= d_2 + \frac{3}{4} \tilde{\varepsilon}. \end{aligned}$$

This is a contradiction and therefore there exists $t \in [0, t_\infty]$ with $\chi(t, u) \notin B_{r^*}$ or $\chi(t, u) \notin \rho_{d_1}^{d_2}$. If the second case is true we find

$$(13) \quad \rho(\chi(t_\infty, u)) > d_2.$$

If $\chi(t', u) \notin B_{r^*}$ for some $t' \in [0, t_\infty]$ we may assume

$\rho(\chi(t_\infty, u)) \leq d_2$ (otherwise (13) holds). Define

$$t_2 = \inf \{t \in [0, t_\infty] \mid \chi(t, u) \notin B_{r^*}\} \quad \text{and}$$

$$t_1 = \sup \{t \in [0, t_2] \mid \chi(t, u) \in B_r\}. \quad \text{Clearly } 0 < t_1 < t_2$$

since $r < r^*$ and $U_2 \subset \overset{\circ}{B}_r$. We obtain

$$\begin{aligned}
 (14) \quad \rho(\chi(t_\infty, u)) &\geq d_1 + \frac{3}{4} \tilde{\varepsilon} + \int_{t_1}^{t_2} \frac{d}{dt} \rho(\chi(t, u)) \, dt \\
 &\geq d_1 + \frac{3}{4} \tilde{\varepsilon} + \frac{\varepsilon}{4} (t_2 - t_1) .
 \end{aligned}$$

Let us estimate $t_2 - t_1$. By Proposition 1 (iii) we know that

$$\|\chi(t_2, u)\| \leq (\|\chi(t_1, u)\| + 1) e^{t_2 - t_1} .$$

$\|\chi(t_2, u)\| = r^*$ and $\|\chi(t_1, u)\| = r$ we deduce

$$\ln(r^*/(r+1)) \leq t_2 - t_1 .$$

By the definition of r^* we infer $\frac{4}{\varepsilon} (d_2 - d_1) \leq t_2 - t_1$.

Combining this with (14) yields

$$(15) \quad \rho(\chi(t_\infty, u)) \geq d_2 + \frac{3}{4} \tilde{\varepsilon} .$$

Hence we conclude, since $u \in U_2$ was arbitrarily chosen, that

$$\chi(t_\infty, U_2) \subset \rho_{d_2} .$$

This implies a contradiction by (12):

$$d_2 \leq \inf \rho(\chi(t_\infty, U_2)) \leq C(\Psi) \leq d_2 - \frac{3}{4} \tilde{\varepsilon} .$$

(ii) is proved. Part (i) is easier because one has not to introduce the constants r and r^* and follows from the intersection hypothesis and Corollary 2 using arguments similar to those in the proof of (ii).

□

IV. Proof of the Main Results

Proof of Lemma 1. (i) The convexity of Θ is trivial.

(ii) Let $h \in \text{int } \Theta$. We find $\delta_0 > 0$ such that $\bar{h} \in \text{int } \Theta$ for all $\bar{h} \in H$, $\|h - \bar{h}\| \leq \delta_0$. This implies for all $y \in Y$

$$\liminf_{t \rightarrow +\infty} (b(ty)/t + (\bar{h}, y)) \geq 0.$$

For given $y \in Y$, $\|y\| = 1$, taking $\bar{h}$ such that $(\bar{h} - h, y) = -\delta_0$ we deduce making use of the homogeneity of J_b

$$J_b(y) + (h, y) \geq \delta_0 \|y\|.$$

This shows $J_b(y) > -(h, y)$ for all $y \in Y \setminus \{0\}$. Conversely, if $J_b(y) > -(h, y)$ for all $y \in Y \setminus \{0\}$ we deduce by the continuity of J_b and the compactness of $(\partial B_1) \cap Y$ the existence of a constant $\delta_0 > 0$ with

$$J_b(y) + (h, y) \geq \delta_0 \|y\| \quad \text{for all } y \in Y.$$

Now let $\bar{h} \in H$, $\|h - \bar{h}\| \leq \delta_0/2$. We show that $\bar{h} \in \Theta$ which completes the proof of (ii). We have $J_b(y) + (\bar{h}, y) \geq (\delta_0/2) \|y\|$ for all $y \in Y$. Let $(y_n) \subset Y$ be an arbitrary sequence with $\|y_n\| \rightarrow +\infty$. Since $\dim Y < +\infty$ we may assume that $\lim y_n / \|y_n\| = v$ exists. Let $t_n = \|y_n\|$. We may assume that $t_n > 0$ for all $n \in \mathbb{N}$ and find for $v_n = y_n / \|y_n\|$

$$\begin{aligned} & b(t_n v_n) / t_n - J_b(v_n) \\ &= b(t_n v) / t_n - J_b(v_n) + b(t_n v_n) / t_n - b(t_n v) / t_n \\ &\geq b(t_n v) / t_n - J_b(v_n) - \|B\| \|v_n - v\|. \end{aligned}$$

Therefore

$$\begin{aligned}
 & \liminf_{n \rightarrow +\infty} (b(t_n v_n)/t_n - J_b(v_n)) \\
 & \geq \liminf_{t \rightarrow \infty} (b(t v_n)/t_n) - J_b(v) \\
 & \geq \liminf_{t \rightarrow +\infty} b(tv)/t - J_b(v) \\
 & = J_b(v) - J_b(v) \\
 & = 0 .
 \end{aligned}$$

Hence

$$\begin{aligned}
 b(y_n) + (\bar{h}, y_n) &= J_b(y_n) + (\bar{h}, y_n) + b(y_n) - J_b(y_n) \\
 &\geq (\delta_0/2) \|y_n\| + ((b(y_n) - J_b(y_n))/\|y_n\|) \|y_n\| \\
 &= (\frac{\delta_0}{2} + \varepsilon_n) \|y_n\| \rightarrow +\infty ,
 \end{aligned}$$

$$\text{where } \liminf \varepsilon_n = \liminf (b(y_n)/\|y_n\| - J_b(\frac{y_n}{\|y_n\|})) \geq 0 .$$

This proves (ii).

(iii) If $h \in \text{cl}(\Theta)$ there exists for given $\varepsilon > 0$, $\bar{h} \in \Theta$ with $\|h - \bar{h}\| < \varepsilon$. We find for $y \neq 0$

$$\liminf_{t \rightarrow +\infty} (b(ty) + (\bar{h}, ty)) = +\infty$$

Hence

$$\begin{aligned}
 J_b(y) &\geq -(\bar{h}, y) \\
 &= -(h, y) + (h - \bar{h}, y) \\
 &\geq -(h, y) - \varepsilon \|y\| .
 \end{aligned}$$

Since $\varepsilon > 0$ is arbitrarily given our assertion follows.

(iv) Let $(y_n) \subset Y$ be a sequence with $\|y_n\| \rightarrow +\infty$. If $h \in \tilde{\Theta} \equiv \{h \in H \mid -(h, y) \leq J_b(y) \text{ for all } y \in Y\}$ we have

$$J_b(y_n) + (h, y_n) \geq 0 \quad \text{for all } n \in \mathbb{N}.$$

This implies

$$\begin{aligned} b(y_n) + (h, y_n) &= b(y_n) - J_b(y_n) + J_b(y_n) + (h, y_n) \\ &\geq b(y_n) - J_b(y_n) \rightarrow +\infty. \end{aligned}$$

Hence $h \in \Theta$. By (iii) we conclude therefore $\Theta \subset \text{cl}(\tilde{\Theta}) \subset \tilde{\Theta} \subset \Theta$ which implies $\Theta = \text{cl}(\tilde{\Theta})$ and $\Theta = \{h \in H \mid -(h, y) \leq J_b(y) \text{ for all } y \in Y\}$.

□

Proof of Theorem 1

For $\tilde{h} \in \Theta$ we first construct a pair of intersecting sets and show that $\phi_{\tilde{h}}$ satisfies (GPS). Perturbation Theorem 6 (ii) then yields a first solution, also for h in a neighbourhood $V(\tilde{h})$ of $\tilde{h}$. For fixed $h \in V(\tilde{h}) \setminus \Theta$ we construct a second pair of intersecting sets. We can not prove that ϕ_h satisfies (GPS); however this condition holds for very special perturbations of ϕ_h ; which allow passage to the limit.

Step 1. For given $\tilde{h} \in \Theta$ there exists $\delta > 0$ such that

$U_1 = \{x+y \in X \oplus Y \mid \|x+y\| = \delta\}$ verifies

$$\inf \phi_{\tilde{h}}(U_1) > \sup \phi_{\tilde{h}}(Z).$$

This follows immediately from the following estimates

$$\begin{aligned}\phi_{\tilde{h}}^{\sim}(z) &= a(z) + b(z) + \langle \tilde{h}, z \rangle \\ &\quad - \alpha \|z\|^2 + (\|B\| + \|\tilde{h}\|) \|z\| \quad \text{for all } z \in Z\end{aligned}$$

and

$$\begin{aligned}\phi_{\tilde{h}}^{\sim}(x+y) &= a(x) + b(x+y) + \langle \tilde{h}, x+y \rangle \\ &\quad - \alpha \|x\|^2 - (\|B\| + \|\tilde{h}\|) \|x\| + b(y) + \langle \tilde{h}, y \rangle\end{aligned}$$

because $\tilde{h} \in \Theta$.

Step 2. $\phi_{\tilde{h}}^{\sim}$ verifies the (GPS)-condition on bounded intervals ($h \in \cdot$).

Assume $(\phi_{\tilde{h}}^{\sim}(u_n))$ is bounded and $\phi_{\tilde{h}}^{\sim}(u_n) \rightarrow 0$. We immediately infer that $((I-P)u_n)$ is bounded. Hence, if $\|u_n\| \rightarrow +\infty$ we must have $\|Pu_n\| \rightarrow +\infty$ and find since $a(u_n) = a((I-P)u_n)$ is bounded

$$\begin{aligned}+\infty &> \limsup \phi_{\tilde{h}}^{\sim}(u_n) \\ &\geq \text{const} + \limsup (b(Pu_n) + \langle \tilde{h}, Pu_n \rangle) \\ &= +\infty\end{aligned}$$

since $\tilde{h} \in \Theta$. This contradiction proves the boundedness of (u_n) .

Step 3. Assume $\tilde{h} \in \Theta$ is given and U_1 is as in step 1. Let

$U_2 = \{x+y \in X \oplus Y \mid \|x+y\| \leq \delta\}$ and $\tilde{\varepsilon} = \frac{1}{4} (\inf \phi_{\tilde{h}}^{\sim}(U_1) - \sup \phi_{\tilde{h}}^{\sim}(Z)) (>0)$. Then there exists an open neighbourhood $V(\tilde{h})$

such that $\phi_{\tilde{h}}$ possesses for all $h \in V(\tilde{h})$ a critical point $u \in H$

with $\phi_{\tilde{h}}(u) \in [d_1, d_2]$ where $d_1 = \inf \phi_{\tilde{h}}^{\sim}(U_2) - \tilde{\varepsilon}$, $d_2 = \sup \phi_{\tilde{h}}^{\sim}(Z) + \tilde{\varepsilon}$.

Moreover $\inf \phi_{\tilde{h}}^{\sim}(U_1) \geq d_2 + 2\tilde{\varepsilon}$ and $\sup \phi_{\tilde{h}}^{\sim}(Z) \leq d_2$ for all $h \in V(\tilde{h})$.

In order to apply Theorem 6 define if P_B is compact

$$\tilde{M} = \{(\eta, \eta_0) \in M^*(A, P_- + P) \mid \eta = \hat{\eta} \mid [0, 1] \times H, \hat{\eta} \in M(A, P_- + P, \mathbb{R} \times H) \\ \text{is a flow with } \hat{\gamma}(0, u) = \text{Id}, \hat{K}(0, u) = 0\}$$

if $P_+ B$ is compact let

$$\tilde{M} = \{(\eta, \eta_0) \in M^*(A, P_+) \mid \eta = \hat{\eta} \mid [0, 1] \times H, \hat{\eta} \in M(A, P_+, \mathbb{R} \times H) \\ \text{is a flow with } \hat{\gamma}(0, u) = \text{Id}, \hat{K}(0, u) = 0\}.$$

We find $r > 0$ such that $V_2 = \{z \in Z \mid \|z\| \leq r\}$ and $V_1 = \partial_Z V_2$ verify together with U_1, U_2 defined before

$$\inf \phi_{\tilde{h}}^{\sim}(U_1) > \sup \phi_{\tilde{h}}^{\sim}(V_2)$$

$$\inf \phi_{\tilde{h}}^{\sim}(U_2) > \sup \phi_{\tilde{h}}^{\sim}(V_1)$$

and moreover $(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$ by example 1 or example 2. By Theorem 6 (ii) we find a neighbourhood $\hat{V}(\tilde{h})$ such that for all $h \in \hat{V}(\tilde{h})$ there exists a bounded sequence $(u_n) \subset (\phi_h)_{d_1}^{d_2}$ with $\phi'_h(u_n) \rightarrow 0$. We may assume $\phi_h(u_n) \rightarrow d$ for some $d \in [d_1, d_2]$

and find by our hypotheses that d is a critical value for ϕ_h .

By the definition of $\tilde{c}$ we have $\inf \phi_{\tilde{h}}^{\sim}(U_1) = \sup \phi_{\tilde{h}}^{\sim}(Z) + 4\tilde{c} = d_2 + 3\tilde{c}$

Note that $\sup \phi_h(Z) = \sup \{\phi_h(z) \mid \|z\| \leq c, z \in Z\}$ for all $h \in H$, $\|h - \tilde{h}\| \leq 1$, with a suitable constant $c > 0$ (this follows from an estimate similar to that given in step 1). Since U_1 is bounded, we find an open neighbourhood $V(\tilde{h}) \subset B_1$ such that

$$\inf \phi_h(U_1) \geq d_2 + 2\tilde{c}, \text{ and}$$

$$\sup \phi_h(Z) = d_2.$$

Let $V(\tilde{h}) = \tilde{V}(\tilde{h}) \cap \hat{V}(\tilde{h})$. By construction $V(\tilde{h})$ has the desired properties.

Step 4. Let $U = \bigcup_{\tilde{h} \in \Theta} V(\tilde{h})$ where $V(\tilde{h})$ is the set constructed in step 3. If

$h \in U \setminus \Theta$ ϕ_h possesses at least two critical points.

By construction of U we have for all $h \in U$ at least one critical point. If $h \in U \setminus \Theta$ there exists $\tilde{h} \in \Theta$ with $h \in V(\tilde{h}) \setminus \Theta$.

Let U_1 , d_2 and $\tilde{\varepsilon} > 0$ be as in step 3. We have

$$\begin{aligned} \sup_{h \in U} \phi_h(z) &\leq d_2 \\ \inf_{h \in U_1} \phi_h(U_1) &\geq d_2 + 2\tilde{\varepsilon} \end{aligned}$$

and ϕ_h possesses a critical point u' with $\phi_h(u') \leq d_2$. Now we show that ϕ_h possesses a critical point u'' with $\phi_h(u'') \geq d_2 + \tilde{\varepsilon}$, which completes the proof. By the characterisation of Θ given in Lemma 1 we find $y_0 \in Y$, $\|y_0\| = 1$ with $J_b(y_0) < -(h, y_0)$, or equivalently $\liminf_{t \rightarrow +\infty} (b(ty_0) + (h, ty_0)) = -\infty$. We have

for $u = z + ty_0$, $z \in Z$, $t > 0$:

$$\begin{aligned} (16) \quad \phi_h(u) &\leq -\alpha \|z\|^2 + (\|B\| + \|h\|) \|z\| + b(ty_0) + (h, ty_0) \\ &\leq c_2 + b(ty_0) + (h, ty_0) \end{aligned}$$

where c_2 is a constant independent of $t \in \mathbb{R}^+$ and $z \in Z$. Moreover $\sup_{h \in U} \phi_h(z) \leq d_2$. We find a constant $t_0 > \delta$ (where δ is the constant defining U_1) such that

$$b(t_0 y_0) + (h, t_0 y_0) \leq d_2 - c_2.$$

Let $c_3 = \sup_{t \in [0, t_0]} (b(ty_0) + (h, ty_0))$. Since $-\alpha \|z\|^2 + (\|B\| + \|h\|) \|z\| \leq -c_3 + d$

for all $z \in Z$, $\|z\| \geq \rho_1 > 0$ for a suitably large constant ρ_1 ,

we deduce for $V = [0, t_0]y_0 \oplus \{z \in Z \mid \|z\| \leq \rho_1\}$ that

$$\sup \phi_h(\partial V) \leq d_2 \quad (\text{boundary taken in } \mathbb{R}y_0 \oplus Z)$$

making use of (16). Let $U'_1 = \emptyset$, $U'_2 = U_1$, $V'_1 = \partial V$, $V'_2 = V$ and $\tilde{M}$ defined as in step 3. We have $(U'_1, U'_2) \overset{M}{\sim} (V'_1, V'_2)$ by example 3 or 4 and

$$\begin{aligned} \inf \phi_h(U'_2) &> \sup \phi_h(V'_1) \\ + \infty &= [\inf \phi_h(U'_1)] > \sup \phi_h(V'_2). \end{aligned}$$

Applying Theorem 6 (i) we find $\varepsilon_0 > 0$ such that for all

functionals $\phi_{h, \varepsilon} \in C^1(H, \mathbb{R})$, $\phi_{h, \varepsilon}(u) = \phi_h(u) - \frac{\varepsilon}{2} \|Pu\|^2$,

$\varepsilon \in (0, \varepsilon_0]$, there exists a sequence $(u_n^\varepsilon) \subset H$ with

$(\phi_{h, \varepsilon}(u_n^\varepsilon)) \subset [d_2 + \varepsilon, d']$ and $\phi_{h, \varepsilon}'(u_n^\varepsilon) \rightarrow 0$, $n \rightarrow \infty$, where

$d' = \sup \phi_h(V'_2) + 1$. Let $\varepsilon_k = \frac{1}{k}$. For k large enough we have

$\varepsilon_k \in (0, \varepsilon_0]$. We find $n(k) \in \mathbb{N}$ such that $\|\phi_{h, \varepsilon_k}'(u_k)\| \leq 1$, $u_k = u_{n(k)}^{\varepsilon_k}$,

and

$$(\phi_{h, \varepsilon_k}'(u_k), u_k) \leq \frac{k}{4} \|u_k\|^4.$$

Observing that $(\phi_{h, \varepsilon_k}(u_k))$ is bounded we find

$$\begin{aligned}
b(u_k) - (Bu_k, u_k) &= (a(u_k) + b(u_k) + (h, u_k) - \frac{\epsilon_k}{2} \|Pu_k\|^2) \\
&\quad - (2a(u_k) + (Bu_k, u_k) + (h, u_k) - \epsilon_k \|Pu_k\|^2) \\
&\quad + a(u_k) - \frac{\epsilon_k}{2} \|Pu_k\|^2 \\
&= \phi_{h, \epsilon_k}(u_k) - (\phi'_{h, \epsilon_k}(u_k), u_k) \\
&\quad + a((I-P)u_k) - \frac{\epsilon_k}{2} \|Pu_k\|^2 \\
&\quad \cdot d' + \frac{\epsilon_k}{4} \|u_k\|^2 - \frac{\epsilon_k}{2} \|Pu_k\|^2 \\
&\quad + a((I-P)u_k) \\
&\leq d' - \frac{\epsilon_k}{4} \|Pu_k\|^2 + \text{const.} \\
&\leq d''
\end{aligned}$$

for a suitable constant $d'' < +\infty$ since it is clear that

$(\|(I-P)u_n\|)$ and $(a((I-P)u_k))$ are uniformly bounded because $\|\phi'_{h, \epsilon_k}(u_k)\| \leq 1$. Now if $\|u_k\| \rightarrow +\infty$ we infer that $(\|(I-P)u_k\|)$ is bounded and $\|Pu_k\| \rightarrow +\infty$. By our hypotheses this implies a contradiction:

$$+\infty = \liminf (b(u_k) - (Bu_k, u_k)) \leq d'' < +\infty.$$

Hence (u_k) is bounded. Therefore eventually taking a subsequence if necessary

$$\begin{aligned}
\phi'_h(u_k) &\rightarrow 0 \quad \text{and} \\
\phi_h(u_k) &\rightarrow \hat{d} \in [d_2 + \tilde{\epsilon}, d'] .
\end{aligned}$$

By our hypotheses this implies that $\hat{d}$ is a critical value. The proof is complete. $\square$

In order to prove Theorem 2 let us observe that the compactness of P_+B implies that $b|x + Z$ is weakly sequentially continuous for all $x \in X$. For the convenience of the reader we present the short proof given in [10, Th. 8.2, p. 76]. Assume

$x+z_n \rightarrow x+z$ and $b(x+z_n) \not\rightarrow b(x+z)$. We may assume

$|b(x+z_n) - b(x+z)| \geq \varepsilon > 0$ for all $n \in \mathbb{N}$. For all $n \in \mathbb{N}$ we find $t_n \in (0,1)$ with

$$\begin{aligned} b(x+z_n) - b(x+z) &= (B(x+t_n(z_n-z)), z_n-z) \\ &= (P_+B(x+t_n(z_n-z)), z_n-z). \end{aligned}$$

Taking a subsequence we may assume by the compactness of P_+B that $y_n = P_+B(x+t_n(z_n-z)) \rightarrow y$. Hence

$$\varepsilon \leq \liminf |b(x+z_n) - b(x+z)| = \liminf |(y_n, z_n-z)| = 0.$$

Proof of Theorem 2

For $\tilde{h} \in X \oplus Z$ we construct a pair of intersecting sets and show that $\phi_{\tilde{h}}$ satisfies (GPS) on compact sets of $\mathbb{R} \setminus \{c_{\tilde{h}}\}$ where the constant $c_{\tilde{h}}$ will be made precise in the proof. Theorem 6 (i) gives a first solution, also for $\phi_{\tilde{h}}$ with $\tilde{h}$ in a neighbourhood $V(\tilde{h})$.

For $h \in V(\tilde{h}) \setminus (X \oplus Z)$, a second pair of intersecting sets is constructed. Since $\phi_{\tilde{h}}$ satisfies (GPS), the existence of a second solution follows from Theorem 6 (i).

Let $A^{-1}: X \oplus Z \rightarrow X \oplus Z$ be the generalized inverse of A . For $h \in H$, set $a_h(u) = a(u) + (h, u)$.

Step 1. For all $h \in X \oplus Z$ setting $x_0 = -A^{-1}P_-h$, $z_0 = -A^{-1}P_+h$ there exists an open neighbourhood $V(\tilde{h})$ such that

$$\sup \phi_h(x_0+Z) < c_h + \Delta \quad \text{for all } h \in V(\tilde{h}),$$

$$\text{with } c_h = a_h(-A^{-1}(I-P)h).$$

Moreover there exists a constant $r_0 > \|x_0\|$ such that, with

$$\Lambda = (X \oplus Y) \cap \partial B_{r_0} \quad \text{we have}$$

$$\inf \phi_h(z_0+\Lambda) > \sup \phi_h(x_0+Z).$$

for all $h \in V(\tilde{h})$.

Fix $\tilde{h} \in X \oplus Z$ and let $c_- = a_{\tilde{h}}(x_0)$ and $c_+ = a_{\tilde{h}}(z_0)$.

Note that at z_0 $a_{\tilde{h}}|Z$ attains its maximum and that z_0 is the (unique) solution of $Az + P_+\tilde{h} = 0$ in Z . Moreover

$\sup \phi_{\tilde{h}}(x_0+Z) = \sup\{\phi_{\tilde{h}}(x_0+z) \mid \|z\| \leq r_1\}$ for a suitably large constant $r_1 > 0$. We find

$$\begin{aligned} \sup \phi_{\tilde{h}}(x_0+Z) &= a_{\tilde{h}}(x_0) + \sup_Z (a_{\tilde{h}}(z) + b(x_0+z)) \\ &= c_- + \sup_{\|z\| \leq r_1} (a_{\tilde{h}}(z) + b(x_0+z)) \\ &\leq c_- + \sup a_{\tilde{h}}(Z) + \sup_{\|z\| \leq r_1} b(x_0+z) \\ &\equiv c_- + c_+ + \bar{b} \end{aligned}$$

where $\bar{b} = \sup_{\|z\| \leq r_1} b(x_0+z)$. There exists a sequence $(z_n) \subset Z$,

$\|z_n\| \leq r_1$, $z_n \rightarrow z$ such that

$$\bar{b} = \lim b(x_0 + z_n) = b(x_0 + z) < \Delta.$$

Hence

$$(17) \quad \sup \phi_h^-(x_0 + z) \leq c_- + c_+ + \bar{b} < c_- + c_+ + \Delta.$$

Next we show that

$$(18) \quad \sup \phi_h^-(x_0 + z) < \inf \phi_h^-(z_0 + \Lambda)$$

for a suitable constant $r_0 > \|x_0\|$ ($\Lambda = \Lambda(r_0)$). If not there exists a sequence $(x_n + y_n) \subset X \oplus Y$ with $\|x_n + y_n\| \rightarrow +\infty$ such that

$$\phi_h^-(z_0 + x_n + y_n) \leq \sup \phi_h^-(x_0 + z) < c_- + c_+ + \bar{b}.$$

If $\|x_n\| \rightarrow +\infty$ one immediately obtains a contradiction. Therefore we may assume $\|y_n\| \rightarrow +\infty$ and (x_n) bounded. By our hypotheses this yields

$$c_+ + \inf_n a_h^-(x_n) + \lim b(z_0 + x_n + y_n)$$

$$\lim \inf \phi_h^-(z_0 + x_n + y_n) \leq c_- + c_+ + \bar{b}.$$

Observing that $c_- = \inf a_h^-(X) \leq \inf_n a_h^-(x_n)$ we find

$$\lim \inf \phi_h^-(z_0 + x_n + y_n) \leq c_- + c_+ + \bar{b} < c_- + c_+ + \Delta.$$

This contradiction proves (18) for some constant $r_0 > \|x_0\|$. Since there exists a constant $c > 0$ such that for all $h \in H$,

$\|h - \tilde{h}\| \leq 1$ we have

$$\sup \phi_{\tilde{h}}(x_0 + Z) = \sup \{ \phi_{\tilde{h}}(x_0 + Z) \mid \|z\| \leq c, z \in Z \}$$

we find a neighbourhood $V(\tilde{h}) \subset B_1(\tilde{h})$ such that (using (17), (18) and the continuity of $h \rightarrow c_h$, $c_{\tilde{h}} = c_+ + c_-$)

$$\sup \phi_{\tilde{h}}(x_0 + Z) < c_{\tilde{h}} + \Delta, \text{ and}$$

$$\inf \phi_{\tilde{h}}(z_0 + \Lambda) > \sup \phi_{\tilde{h}}(x_0 + Z) \text{ for all } h \in V(\tilde{h}).$$

Then for each $\tilde{h} \in X \oplus Z$ we can construct a set $V(\tilde{h})$ having the properties stated in step 1. Define

$$U = \bigcup_{\tilde{h} \in X \oplus Z} V(\tilde{h})$$

Step 2. Let $\tilde{h} \in X \oplus Z$ be given and let $V(\tilde{h})$, x_0, z_0, Λ , as constructed in step 1.

(i) For $h \in V(\tilde{h})$ let

$$\varepsilon_h = \min \{ (\inf \phi_h(z_0 + \Lambda) - \sup \phi_h(x_0 + Z)) / 4, (c_h + \Delta - \sup \phi_h(x_0 + Z)) / 4 \}.$$

Then there exists for given $h \in V(\tilde{h})$ at least one critical point

$$u_1 \text{ with } \phi_h(u_1) \leq \sup \phi_h(x_0 + Z) + \varepsilon_h < c_h + \Delta.$$

(ii) If $h \in V(\tilde{h}) \setminus X \oplus Z$ there exists a second critical point u_2

$$\text{with } \phi_h(u_2) \geq \inf \phi_h(x_0 + \Lambda) - \varepsilon_h. (> \phi_h(u_1)).$$

(i) Let $h \in V(\tilde{h})$ be given. Define $U_1 = z_0 + \Lambda$,

$U_2 = z_0 + \{x+y \in X \oplus Y \mid \|x+y\| \leq r_0\}$. Moreover there exists a

constant $r' > \|z_0\|$ such that $\sup \{\phi_h(x_0+z) \mid \|z\| = r'\} < \inf \phi_h(U_2)$.

Let $V_1 = x_0 + (Z \cap \partial B_r)$ and $V_2 = x_0 + (Z \cap B_r)$. We have

$$\inf \phi_h(U_1) > \sup \phi_h(V_2) \quad (\text{step 1}),$$

$$\inf \phi_h(U_2) > \sup \phi_h(V_1),$$

$$\sup \phi_h(V_2) < c_h + \Delta. \quad (\text{step 1}).$$

Let $\tilde{M} = \{(\eta, \eta_0) \in M^*(A, P_+) \mid \eta = \hat{\eta} \mid [0, 1] \times H, \hat{\eta} \in M(A, P_+, \mathbb{R} \times H) \\ \text{is a flow with } \hat{\gamma}(0, u) = \text{Id}, \hat{K}(0, u) = 0\}$

We find by example 2 and the remark that

$$(U_1, U_2) \overset{M}{\sim} (V_1, V_2)$$

Let $d_1 = \inf \phi_h(U_2) - \varepsilon_h$, $d_2 = \sup \phi_h(V_2) + \varepsilon_h$ ($< c_h + \Delta$).

Now applying Theorem 6(i) we find a sequence (u_n) with

$\phi'_h(u_n) \rightarrow 0$ and $\phi_h(u_n) \rightarrow d \in [d_1, d_2]$. Clearly $((I-P)u_n)$ must be bounded. Hence, if $\|u_n\| \rightarrow +\infty$ we must have $\|Pu_n\| \rightarrow +\infty$ and

find

$$\begin{aligned} 0 &= \lim \phi'_h(u_n) \\ &= \lim (Au_n + Bu_n + h) \\ &= \lim (A(I-P)u_n) + h \end{aligned}$$

if $Ph \neq 0$ we have a contradiction since $R(A) = X \oplus Z = Y^\perp$.

Therefore we may assume $Ph = 0$ or equivalently $h \in X \oplus Z$.

$A : X \oplus Z \rightarrow X \oplus Z$ is a topological isomorphism. Therefore

$(I-P)u_n \rightarrow -A^{-1}h$. We find

$$\begin{aligned}
d &= \lim \phi_h(u_n) \\
&= \lim a_h((I-P)u_n) + \lim b(u_n) \\
&= a_h(-A^{-1}h) + \Delta \\
&= c_h + \Delta > d.
\end{aligned}$$

Therefore (u_n) must be bounded. By our hypotheses there exists u_1 with $\phi_h(u_1) = d$ and $\phi_h'(u_1) = 0$.

(ii) $h \in V(\tilde{h}) \setminus X \oplus Z$ let $U_1' = \emptyset$ and $U_2' = U_1$. Let $e = -Ph/\|Ph\|$.

For $u = x_0 + z + te$, $z \in Z$, $t \in \mathbb{R}^+$ we have

$$(19) \quad \phi_h(u) = a_h(x_0) + a_h(z) + b(x_0 + z + te) - t\|Ph\|.$$

Hence we find $t_0 > 2r_0$ such that

$$(20) \quad \sup \phi_h(x_0 + z + t_0 e) = \sup \phi_h(x_0 + z).$$

Moreover from the estimate (19) we infer that, for a number $r'' > \|z_0\|$ large enough,

$$\sup \{ \phi_h(x_0 + z + te) \mid t \in [0, t_0], \|z\| = r'' \} \leq \sup \phi_h(x_0 + z).$$

This proves that $V_2' = x_0 + ([0, t_0]e \oplus Z \cap B_{r''})$, $V_1' = \partial V_2'$

(∂ taken in $x_0 + \mathbb{R}e \oplus Z$) verify

$$\sup \phi_h(V_1') \leq \sup \phi_h(x_0 + z) < \inf \phi_h(U_1) = \inf \phi_h(U_2'). \quad (\text{Step 1})$$

Clearly

$$\sup \phi_h(V_2') < +\infty [= \inf \phi_h(U_1')].$$

Let $\tilde{M}$ be as above. By example 4 and the remark we have

$$(U_1', U_2') \overset{M}{\sim} (V_1', V_2')$$

and applying Theorem 6(i) we obtain a sequence (u_n') with

$$\phi_h(u_n') \rightarrow d' \geq \inf \phi_h(U_2') - \varepsilon_h = \inf \phi_h(U_1') - \varepsilon_h \quad \text{and} \quad \phi_h'(u_n') \rightarrow 0.$$

Since $Ph \neq 0$, the argument used before in part (i) shows that

(u_n') must be bounded. Hence d' is a critical value.

Step 3. $\hat{U}$ is closed.

Let $h \in cl(\hat{U})$. We have to show that $h \in \hat{U}$. Since $X \oplus Z \subset \text{int } \hat{U}$

by step 2 we may assume $h \notin X \oplus Z$. We find a sequence

$$(h_n) \subset \hat{U} \text{ with } h_n \rightarrow h \text{ and a corresponding sequence } (u_n) \subset H$$

such that $\phi_{h_n}'(u_n) = 0$. Since $Ph \neq 0$ we find as in the proof

of step 2 that (u_n) is bounded. This implies the boundness of

$(\phi_{h_n}(u_n))$ and we may assume $\phi_{h_n}(u_n) \rightarrow c$ for a suitable number.

Hence

$$\phi_h(u_n) \rightarrow c$$

$$\phi_h'(u_n) \rightarrow 0$$

for a bounded sequence (u_n) and the result follows.

□

In order to reduce the wave equation problem to an abstract problem to which Theorems 1 and 2 are applicable denote by

$(E_\lambda)_{\lambda \in \mathbb{R}}$ the spectral resolution of L and define orthogonal projection by

$$P_- = \int_{-\infty}^{0-} dE_\lambda, \quad P = \int_{0-}^{0+} dE_\lambda, \quad P_+ = \int_{0+}^{+\infty} dE_\lambda,$$

and corresponding mutually orthogonal subspaces X, Y, Z by

$$X = P_- H, \quad Y = P H, \quad Z = P_+ H.$$

Clearly $H = X \oplus Y \oplus Z$. Define bounded linear selfadjoint operators by

$$R = \int_{-\infty}^{0-} (-\lambda)^{-1/2} dE_\lambda, \quad S = \int_{0+}^{+\infty} \lambda^{-1/2} dE_\lambda.$$

Since 0 is an isolated point of the spectrum both operators are well-defined and bounded. Moreover S is compact because $\sigma(L) \cap \mathbb{R}^+$ contains only in $\mathbb{R}$ isolated points having finite multiplicity. Define $\Phi \in C^1(H, \mathbb{R})$ by

$$(21) \quad \begin{aligned} \Phi(u) &\equiv a(u) + b(u) \\ &\equiv \frac{1}{2} ((P_- - P_+)u, u) + f((R + P + S)u) \end{aligned}$$

and let $\bar{\Phi}_h = \Phi + ((R + P + S)h, \cdot) = \Phi_{(R + P + S)h}$. If

u is a critical point of $\bar{\Phi}_h$ we find

$$(P_- - P_+)u + (R + P + S)(F(R + P + S)u + h) = 0.$$

Applying $-R + P + S$ we obtain for $w = (R + P + S)u$

$$-(I-P)w + (S^2 - R^2 + P)(Fw + h) = 0.$$

Observing that $S^2 - R^2 + P = (L + P)^{-1}$ we find

$$Lw - Fw = h.$$

Hence, in order to find weak solutions of the wave-equation problem it is enough to study $\bar{\phi}_h$.

Clearly ϕ verifies (ϕ) (see I) and $P_+B = SF(R + P + S)$ is compact since S is compact. We need

Lemma 2. Assume g verifies (F0) and (F1) and let L, B, b be as defined before. Then we have for all sequences $(u_n) \subset H$ with $((I-P)u_n)$ bounded, $\|Pu_n\| \rightarrow +\infty$:

$$\liminf (b(u_n) - J_b(Pu_n)) = +\infty$$

$$\liminf (b(u_n) - (Bu_n, u_n)) = +\infty$$

and

$$J_b(y) = \lim_{t \rightarrow +\infty} b(ty)/t = \gamma_{+\infty} \int_M Y_+ + \gamma_{-\infty} \int_M Y_- , \quad y \in Y$$

Proof. Let $y \in N(L) \setminus \{0\}$. We have for $t > 0$ $b(ty)/t = \int_M G(ty)/t$ and $|G(ty)|/t \leq \text{const}$. By Lebesgue's Theorem using our hypotheses

$$\begin{aligned} J_b(y) &= \liminf_{t \rightarrow +\infty} b(ty)/t \\ &= \lim_{t \rightarrow +\infty} b(ty)/t \\ &= \int \lim_{t \rightarrow +\infty} G(ty)/t \\ &= \int (\gamma_{+\infty} Y_+ + \gamma_{-\infty} Y_-). \end{aligned}$$

If (u_n) is as in our hypotheses the sequence $(w_n) = ((R + P + S)u_n)$ has the same properties since $Pu_n = Pw_n$. Passing to a subsequence, if necessary, we may assume

$$w_n / \|w_n\| \rightarrow w$$

$$Pw_n / \|w_n\| \rightarrow w$$

$$(w_n / \|w_n\|)(x, t) \rightarrow w(x, t) \quad \text{for a.a. } (x, t) \in M$$

$$(Pw_n / \|w_n\|)(x, t) \rightarrow w(x, t) \quad \text{for a.a. } (x, t) \in M.$$

Since $\|w\| = 1$ and $w \in N(L)$ we infer by the unique continuation property $\text{meas}([w \neq 0] \cap M) = \text{meas}(M)$ and

$$Pw_n(x, t) \rightarrow \begin{cases} +\infty & \text{on } [w > 0] \\ -\infty & \text{on } [w < 0] \end{cases}$$

$$w_n(x, t) \rightarrow \begin{cases} +\infty & \text{on } [w > 0] \\ -\infty & \text{on } [w < 0] \end{cases}$$

Moreover for $v_n = Pw_n$

$$\begin{aligned} b(u_n) - J_b(Pu_n) &= f(w_n) - J_b(Pw_n) \\ &= f(v_n) - J_b(v_n) + f(w_n) - f(v_n) \\ &\geq f(v_n) - J_b(v_n) - \|F\| \sup\{\|(I-P)w_n\| \mid n \in \mathbb{N}\} \\ &= f(v_n) - J_b(v_n) + \text{const.} \end{aligned}$$

We have therefore

$$\begin{aligned}
b(u_n) - J_b(Pu_n) &\geq \int_M ((G(v_n) - \gamma_{+\infty} v_n) \chi_{[v_n > 0]} + (G(u_n) - \gamma_{-\infty} v_n) \chi_{[v_n < 0]}) \\
&\quad - \text{const.} \\
&\equiv \int_M (\alpha_n^+ + \alpha_n^-) - \text{const.}
\end{aligned}$$

By our hypotheses $\alpha_n^+ \rightarrow +\infty$ on $[w > 0]$ and $\alpha_n^- \rightarrow +\infty$ on $[w < 0]$.

By Fatou's Lemma

$$\liminf (b(u_n) - J_b(Pu_n)) = +\infty.$$

Now observe that

$$b(u_n) - (Bu_n, u_n) = f(w_n) - (Fw_n, w_n)$$

and again by Fatou's Lemma

$$\liminf (b(u_n) - (Bu_n, u_n)) = +\infty.$$

□

Lemma 3. Assume g verifies (F0) and (F2) and let L, B, b be as before. Then we have for all sequences $(u_n) \subset H$ with $((I-P)u_n)$ bounded, $\|Pu_n\| \rightarrow +\infty$

$$\lim Bu_n = 0$$

$$\lim b(u_n) = \Delta = \int_M G_\infty = G_\infty \text{ meas}(M)$$

$$b(u) < \Delta \quad \text{for all } u \in H.$$

The proof is left to the reader.

Lemma 4. Let ϕ be as defined in (21) and suppose g verifies (F0). If for a bounded sequence $(u_n) \subset H$, $\phi_h(u_n) \rightarrow d \in \mathbb{R}$,

$\phi'_h(u_n) \rightarrow 0$, then d is a critical value of ϕ_h .

Proof. Let $\chi_n = \phi'_h(u_n)$. We have

$$(22) \quad -\chi_n + (P_- - P_+)u_n + (R + P + S)(F(R + P + S)u_n) + h = 0.$$

Denote by $Q : H \rightarrow N(\square)$ the orthogonal projection. It is easy

to show that $(I - Q)(R + P + S)$ is compact because $\square^{-1} : N(\square)^\perp \rightarrow N(\square)^\perp$ is compact. This implies that $((I - Q)u_n)$ is relatively compact since

$\chi_n \rightarrow 0$ and $\dim N(L) < +\infty$. By the reflexivity of H and

the preceding discussion we may assume

$$\begin{aligned} (I - Q)u_n &\rightarrow u^1 \in N(\square)^\perp \\ Q u_n &\rightarrow u^2 \in N(\square). \end{aligned}$$

Let $w_n = (R + P + S)u_n$ and $w = (R + P + S)(u^1 + u^2)$.

We infer

$$\begin{aligned} (I - Q)w_n &\rightarrow (I - Q)w \\ Q w_n &\rightarrow Qw. \end{aligned}$$

Let $\hat{h} = (-R + P + S)h$. Applying $-R + P + S$ to (22), observing that $-R^2 + S^2 + P = (L + P)^{-1}$ we infer

$$-(-R+P+S)\chi_n - (I-P)w_n + (L+P)^{-1}Fw_n + \hat{h} = 0.$$

This implies

$$(L + P)((I - P)w_n + (-R + P + S)x_n - \hat{h}) = Fw_n.$$

Hence

$$L(w_n + (-R + P + S)x_n - \hat{h}) = Fw_n + P\hat{h} - Px_n.$$

Since the right-hand side is bounded we deduce

$$L(w_n + (-R + P + S)x_n - \hat{h}) \rightharpoonup L(w - \hat{h}).$$

And obviously

$$\square(w_n + (-R + P + S)x_n - \hat{h}) \rightharpoonup \square(w - \hat{h}).$$

Let $\alpha \in H$ be arbitrarily chosen. By the monotonicity of $cI + F$ we find

$$\begin{aligned} & (\square(w_n + (-R + P + S)x_n - \hat{h}), w_n - \alpha) \\ & \geq (c\alpha + F\alpha + P\hat{h} - c\hat{h}, w_n - \alpha) - \text{const. } \|x_n\| \end{aligned}$$

where the constant is only depending on α .

We can take the limit $n \rightarrow +\infty$ observing that

$$(\square(w_n + (-R + P + S)x_n - \hat{h}), w_n - \alpha) = (\square(w_n + (-R + P + S)x_n - \hat{h}), (I - Q)(w_n -$$

Hence

$$(\square(w - \hat{h}), w - \alpha) \geq (c\alpha + F\alpha + P\hat{h} - c\hat{h}, w - \alpha).$$

Now choosing $\alpha = w - t\alpha'$ for $t > 0$ and $\alpha' \in H$ arbitrary we

obtain dividing by t and taking the limit $t \rightarrow 0$

$$(\square(w - \hat{h}) - cw - Fw - P\hat{h} + c\hat{h}, \alpha') \geq 0.$$

Therefore

$$L(w - \hat{h}) - Fw - P\hat{h} = 0.$$

This yields

$$w - \hat{h} - (L + P)^{-1} Fw - Pw = 0.$$

Observing that $w = (R + P + S)(u^1 + u^2)$ and $\hat{h} = (-R + P + S)h$

we obtain by the injectivity of the operator $-R + P + S$ for

$$u_0 = u^1 + u^2$$

$$(P_- - P_+)u_0 + Bu_0 + h = 0.$$

Hence

$$\phi'_h(u_0) = 0.$$

It remains to show that $\phi_h(u_0) = d$.

Since $cI + F$ is monotone and a gradient we have

$$\begin{aligned} & \frac{c}{2} \|w\|^2 + f(w) - \frac{c}{2} \|w_n\|^2 + f(w_n) \\ & \geq (cw_n + Fw_n, w - w_n) \\ & = (\square(w_n + (-R + P + S)\chi_n - \hat{h}) - c((-R + P + S)\chi_n - \hat{h}) - P\hat{h} + P\chi_n, w - w_n). \end{aligned}$$

As before it follows that the right-hand side converges to 0.

Hence

$$\frac{c}{2} \|w\|^2 + f(w) \geq \limsup \left(\frac{c}{2} \|w_n\|^2 + f(w_n) \right).$$

On the other hand by the convexity

$$\frac{c}{2} \|w\|^2 + f(w) \leq \liminf \left(\frac{c}{2} \|w_n\|^2 + f(w_n) \right).$$

Therefore

$$\lim \left(\frac{c}{2} \|w_n\|^2 + f(w_n) \right) = \frac{c}{2} \|w\|^2 + f(w).$$

$$\text{Now since for } \Psi(u) = \frac{1}{2} \|P_- u\|^2 - \frac{1}{2} \|P_+ u\|^2 - \frac{c}{2} \|(R + P + S)u\|^2$$

+ (h, u), u ∈ H we have

$$\Psi(u) = \Psi((I - Q)u) + (h, Qu)$$

we find since $(I - Q)u_n \rightarrow (I - Q)u_0$

$$\begin{aligned} \lim \Psi(u_n) &= \lim \Psi((I - Q)u_n) + \lim (h, Qu_n) \\ &= \Psi((I - Q)u_0) + (h, Qu_0) \\ &= \Psi(u_0) \end{aligned}$$

and therefore

$$\begin{aligned} \lim \phi_h(u_n) &= \lim (\Psi(u_n) + f((R + P + S)u_n) + \frac{c}{2} \|(R + P + S)u_n\|^2) \\ &= \lim \Psi(z_n) + \lim (f(w_n) + \frac{c}{2} \|w_n\|^2) \\ &= \Psi(u_0) + f(w) + \frac{c}{2} \|w\|^2 \\ &= \Psi(u_0) + f((R + P + S)u_0) + \frac{c}{2} \|(R + P + S)u_0\|^2 \\ &= \phi_h(u_0) \end{aligned}$$

which implies $\phi_h(u_0) = d$.

□

Proof of Theorem 3. Lemmas 2 and 4 imply that ϕ verifies the hypotheses of Theorem 1. Hence there exists an open set $\tilde{U}$ containing θ such that (i) and (ii) of Theorem 1 hold. The mapping $\tau : H \rightarrow H : \tau h = (R + P + S)h$ is injective and continuous. Therefore $\tau^{-1}(\tilde{U}) = U$ is open and moreover we have $\tau^{-1}(\theta) = \emptyset$ since $\theta = X \oplus P\theta \oplus Z$ and $\tau P = P\tau = P$.

Let $h \in U$ then $\tau h \in \tilde{U}$ and we find a critical point u_1 of $\phi_{\tau h}$ which is a weak solutions of (1). If $h \in U \setminus \theta$ we have $\tau h \in \tilde{U} \setminus \theta$ and find a second weak solution.

Proof of Theorem 4. Lemmas 3 and 4 imply that ϕ verifies the hypotheses of Theorem 2. As in the proof of Theorem 3 we immediately find that (i) and (ii) hold ((i) is proved for the case $h \in U$). It remains to show (iii). By Theorem 2 (iii) $\hat{\theta} = \{h \in H \mid \phi_h \text{ possesses a critical point}\}$ is closed. Now observe that $\tau^{-1}(\hat{\theta}) = R(L-F)$. Since τ is continuous and $\hat{\theta}$ closed (iii) follows.

□

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V. References

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Lebenslauf

Ich, Helmut, Hermann, Walter Hofer, Deutscher Staatsangehöriger, wurde am 28.2.56 in Sinzig/Rhein (BRD) geboren.

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